Tables of Bessel-Clifford Functions of Orders Zero and One

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Introduction

The present tables are the outgrowth of work performed by the Computation Laboratory for H. K. Skramstad, Assistant Chief, Missile Development Division, National Bureau of Standards. Although the Bessel-Clifford functions are obtainable from existing tables of Bessel functions, it was felt that they warranted tabulation because it is generally necessary to enter the existing tables with an irrational argument. Furthermore, the Bessel-Clifford functions arise as solutions of a class of differential equations occurring in various branches of applied physics, and they are therefore of importance in themselves. A six-place table of the Bessel-Clifford functions $C_0(x)$ and $C_1(x)$ appeared in [3] [1] for $x=0(.02)20$. Since the corresponding second solutions, $D_0(x)$ and $D_1(x)$, were not tabulated heretofore, all four functions are given here to a higher accuracy and over a much wider range of the argument. In addition, the *modified* Bessel-Clifford functions, denoted here by $E_n(x)$ and $G_n(x)$ ($n=0$ and 1) are also tabulated. The present volume carries the tabulation of the functions of orders zero and one up to a point where the asymptotic expansions can be used conveniently.

1. Mathematical Properties

The Bessel-Clifford functions are defined [2] as

$$C_\nu(x)=x^{-\frac{1}{2}\nu}J_\nu(2\sqrt{x}), \quad D_\nu(x)=x^{-\frac{1}{2}\nu}Y_\nu(2\sqrt{x}). \tag{1}$$

They are solutions of the differential equation

$$x\frac{d^2u}{dx^2}+(\nu+1)\frac{du}{dx}+u=0. \tag{2}$$

The *modified* Bessel-Clifford functions are defined by the relations

$$E_\nu(x)=x^{-\frac{1}{2}\nu}I_\nu(2\sqrt{x}), \quad G_\nu(x)=x^{-\frac{1}{2}\nu}K_\nu(2\sqrt{x}). \tag{3}$$

They are solutions of the differential equation

$$x\frac{d^2u}{dx^2}+(\nu+1)\frac{du}{dx}-u=0. \tag{4}$$

Since (4) can be obtained from (2) by substituting $-x$ for x, it is clear that both $C_\nu(-x)$ and $D_\nu(-x)$ are solutions of (4). In fact, $E_\nu(x)=C_\nu(-x)$; and

$$G_\nu(x)=\frac{\pi}{2}e^{i\pi(\nu+1)}[D_\nu(-x)-iC_\nu(-x)].$$

Let $f_\nu(x)$ represent $C_\nu(x)$ or $D_\nu(x)$, and let $g_\nu(x)$ represent $E_\nu(x)$ or $\cos\nu\pi\cdot G_\nu(x)$. (It should be noted that $E_\nu(x)$ and $\cos\nu\pi\cdot G_\nu(x)$ obey the same recurrence relations.[3])

$$f'_\nu(x)=-f_{\nu+1}(x), \quad g'_\nu(x)=g_{\nu+1}(x) \tag{5}$$

$$xf'_\nu(x)-f_{\nu-1}(x)+\nu f_\nu(x)=0, \quad xg'_\nu(x)-g_{\nu-1}(x)+\nu g_\nu(x)=0 \tag{6}$$

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[1] The numbers in brackets refer to items in the References.

[2] The symbol $C_\nu(x)$ is used in [10]; in [5], the functions $C_n(x)$ and $D_n(x)$ are denoted by $F_n(x)$ and $\Gamma_n(x)$, respectively. The symbols C_ν, D_ν, E_ν, and G_ν are used in this Introduction, but they are not perpetuated in the tables.

[3] There is a lack of uniformity in the definition of the Bessel function $K_\nu(t)$. The definition adopted here is that used in [2] and [9]. Whittaker and Watson, in *Modern Analysis*, add the factor $\cos\nu\pi$, so as to have $K_\nu(x)$ obey the same recurrence formulas as $I_\nu(x)$.

$$xf_{\nu+2}(x)-(\nu+1)f_{\nu+1}(x)+f_\nu(x)=0, \quad xg_{\nu+2}(x)+(\nu+1)g_{\nu+1}(x)-g_\nu(x)=0 \tag{7}$$

$$\frac{d}{dx}\{x^{\nu+1}f_{\nu+1}(x)\}=x^\nu f_\nu(x), \quad \frac{d}{dx}\{x^{\nu+1}g_{\nu+1}(x)\}=x^\nu g_\nu(x) \tag{8}$$

$$\frac{d^p}{dx^p}\{x^{\nu+p}f_{\nu+p}(x)\}=x^\nu f_\nu(x), \quad \frac{d^p}{dx^p}\{x^{\nu+p}g_{\nu+p}(x)\}=x^\nu g_\nu(x) \tag{9}$$

$$f_\nu(x+h)=f_\nu(x)-hf_{\nu+1}(x)+\frac{h^2}{2!}f_{\nu+2}(x)-\ldots, \quad g_\nu(x+h)=g_\nu(x)+hg_{\nu+1}(x)+\frac{h^2}{2!}g_{\nu+2}(x)+\ldots. \tag{10}$$

$$\int_0^x f_{-\nu}(t)dt=-f_{-\nu-1}(x), \quad \int_0^x g_{-\nu}(t)dt=g_{-\nu-1}(x). \tag{11}$$

If ν is an integer n, then

$$f_{-n}(x)=(-x)^n f_n(x), \quad g_{-n}(x)=x^n g_n(x) \tag{12}$$

$$\frac{d^{2n}}{dx^{2n}}\{x^n f_n(x)\}=(-1)^n f_n(x), \quad \frac{d^{2n}}{dx^{2n}}\{x^n g_n(x)\}=g_n(x). \tag{13}$$

The power series expansions for $C_\nu(x)$ and $E_\nu(x)$ are

$$C_\nu(x)=x^{-\frac{1}{2}\nu}J_\nu(2\sqrt{x})=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{\Gamma(k+1)\Gamma(\nu+k+1)}, \qquad E_\nu(x)=x^{-\frac{1}{2}\nu}I_\nu(2\sqrt{x})=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)\Gamma(\nu+k+1)}.$$

For positive integral values of n

$$D_n(x)=x^{-\frac{1}{2}n}Y_n(2\sqrt{x})=\frac{1}{\pi}\left[C_n(x)\cdot\log_e x-\sum_{k=0}^{n-1}\frac{(n-k-1)!x^{-n+k}}{k!}-\sum_{k=0}^{\infty}\frac{(-1)^k x^k\{\psi(k+1)+\psi(k+n+1)\}}{k!(k+n)!}\right], \tag{14}$$

and

$$D_0(x)=\frac{1}{\pi}\left[C_0(x)\cdot\log_e(x)-2\sum_{k=0}^{\infty}\frac{(-1)^k x^k\psi(k+1)}{k!k!}\right].$$

in the above

$$\psi(1)=-\gamma; \quad \psi(n)=-\gamma+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}, \quad n\geq 2; \quad \gamma=0.57721\ 56649\ldots \text{ (Euler's constant).}$$

Similarly, if n is a positive integer,

$$G_n(x)=x^{-\frac{1}{2}n}K_n(2\sqrt{x})=[(-1)^{n+1}\tfrac{1}{2}\log_e x]\,E_n(x)+\tfrac{1}{2}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!x^{-n+k}}{k!}$$
$$+(-1)^n\tfrac{1}{2}\sum_{k=0}^{\infty}\frac{x^k\{\psi(k+1)+\psi(k+n+1)\}}{k!(n+k)!} \tag{15}$$

and

$$G_0(x)=-\tfrac{1}{2}E_0(x)\log_e x+\sum_{k=0}^{\infty}\frac{x^k\psi(k+1)}{k!k!}.$$

The asymptotic expansions of the Bessel-Clifford functions are obtainable from the known expressions for Bessel functions. Thus

$$C_n(x)=x^{-\frac{1}{2}n}J_n(2\sqrt{x})\cong(\pi x^{n+\frac{1}{2}})^{-\frac{1}{2}}\left\{P_n(x)\cos\left(2\sqrt{x}-\frac{2n+1}{4}\pi\right)-Q_n(x)\sin\left(2\sqrt{x}-\frac{2n+1}{4}\pi\right)\right\};$$

$$D_n(x)=x^{-\frac{1}{2}n}Y_n(2\sqrt{x})\cong(\pi x^{n+\frac{1}{2}})^{-\frac{1}{2}}\left\{P_n(x)\sin\left(2\sqrt{x}-\frac{2n+1}{4}\pi\right)+Q_n(x)\cos\left(2\sqrt{x}-\frac{2n+1}{4}\pi\right)\right\},$$

where

$$P_n(x)=1+\sum_{k=1}^{\infty}\frac{(-1)^k(4n^2-1^2)(4n^2-3^2)\ldots(4n^2-\overline{4k-1}^2)}{(2k)!2^{8k}x^k},$$

$$Q_n(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(4n^2-1^2)(4n^2-3^2)\ldots(4n^2-\overline{4k-3}^2)}{(2k-1)!\,2^{8k-4}x^{k-\frac{1}{2}}}.$$

Similarly,

$$E_n(x)=x^{-\frac{1}{2}n}I_n(2\sqrt{x})\cong\frac{e^{2\sqrt{x}}}{2(\pi x^{n+\frac{1}{2}})^{\frac{1}{2}}}\left\{1+\sum_{k=1}^{\infty}(-1)^k\phi_k(n,x)\right\},$$
$$G_n(x)=x^{-\frac{1}{2}n}K_n(2\sqrt{x})\cong\frac{e^{-2\sqrt{x}}\sqrt{\pi}}{2(x^{n+\frac{1}{2}})^{\frac{1}{2}}}\left\{1+\sum_{k=1}^{\infty}\phi_k(n,x)\right\}, \tag{16}$$

where

$$\phi_k(n,x)=\frac{(4n^2-1^2)(4n^2-3^2)\ldots(4n^2-\overline{2k-1}^2)}{k!2^{4k}x^{\frac{1}{2}k}}.$$

2. Applications

The following are a few possible uses for the table, which come to mind readily.

(a) The functions tabulated in this volume provide solutions of the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\,y=0, \tag{17}$$

since the general solution of (17) is

$$y=x[AC_1(x)+BD_1(x)].$$

Consequently the functions $C_n(x)$ and $D_n(x)$ will be useful in obtaining an approximate solution of equations of the form $y''+p(x)y=0$ in the neighborhood of the origin when $p(x)$ has a pole there.

(b) The Bessel-Clifford functions are also useful in the solution of the following differential equations [6, p. 146]

$$y''+(1/x)y'+\{(1/x)+(p/2x)^2\}\,y=0,\quad y=Z_p(2i\sqrt{x});$$

$$y''+\{(1/x)-(p^2-1)/4x^2\}\,y=0,\quad y=\sqrt{x}Z_p(2\sqrt{x}),$$

where p is any number and Z_p is a solution of Bessel's differential equation.

(c) In the study of Coulomb wave functions in repulsive fields [11] one must obtain solutions of the differential equation

$$y''+\left\{1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right\}y=0,\quad \eta>0.$$

It can be shown [11] that if η is large, an approximation to the regular solution is

$$y=\rho^{L+1}\Phi_L(\eta,\rho)\sim(2L+1)!\rho^{L+1}E_{2L+1}(2\eta\rho).$$

(d) Laguerre functions occur in various branches of applied mathematics [4]. For large values of n the expansion in terms of Bessel-Clifford functions may be used [8] namely,

$$L_n^{(\alpha)}(t)=\frac{\Gamma(\alpha+n+1)e^{ht}}{n!n^{\alpha}}\sum_{m=0}^{\infty}A_m(t/n)^{\frac{1}{2}(m-\alpha)}J_{\alpha+m}(2\sqrt{nt}),$$

where t and h are real and positive, and the coefficients are defined by the relation

$$\sum_{m=0}^{\infty}A_m z^m=\frac{e^{nz}[1+(h-1)z]^n}{(1+hz)^{\alpha+n+1}}.$$

3. Interpolation

Second central differences, sometimes modified, are tabulated alongside the entries, except for a small region close to the origin, where $D_0(x)$, $D_1(x)$, $G_0(x)$, and $G_1(x)$ have singularities. Wherever differences are given, Everett's interpolation formula, stopping with second differences, can be used to obtain the maximum attainable accuracy. Thus

$$u(x_0+ph)=qu(x_0)+pu(x_1)-E_2\delta_1^2-F_2\delta_0^2. \tag{18}$$

In the above, h is the tabular interval, $x_1=x_0+h$, δ_1^2 and δ_0^2 are the central differences tabulated alongside x_1 and x_0, respectively, and

$$q=1-p, \quad E_2=p(1-p^2)/6, \quad F_2=q(1-q^2)/6.$$

Example. Let it be required to obtain $G_0(x)$ corresponding to $x=6.175225$.

Solution. The following entries are taken from the table:

x	$G_0(x)$	δ^2
6.15	.003857220	2262
6.20	.003773064	2189

Here $x_0=6.15$, $x_1=6.20$, $p=(x-x_0)/h=.5045$, $q=.4955$, $E_2=.06268$, $F_2=.06231$. Applying formula (18),

$$G_0(x)=(.4955)(.003857220)+(.5045)(.003773064)-(.06268)(.000002189)$$
$$-(.06231)(.000002262)=.003814485.$$

In the region where differences are not given, it is best to interpolate in existing tables of $Y_0(x)$, $Y_1(x)$, $K_0(x)$, and $K_1(x)$. The tables in [2] and [9] are particularly suitable, since auxiliary functions are provided there for interpolation close to the origin.

Modified differences, where given, are used exactly like ordinary differences. They are indicated by δ^{2*}, and computed from $\delta^{2*}=\delta^2-0.184\delta^4$. Whenever fifth differences are negligible, and fourth differences are less than 1,000 units in the last place tabulated, the use of δ^{2*} in place of δ^2 reduces the error due to neglecting fourth differences to less than one-half unit of the last place, over and above the usual error due to interpolating in rounded entries. Wherever modified differences are given in this table, the fourth differences can be safely disregarded.

4. Method of Computation and Accuracy of Entries

The values of $C_n(x)$, $D_n(x)$ for all x, and of $E_n(x)$, $G_n(x)$ for $x<100$ were computed by interpolation in [2]. For $x>100$, the asymptotic series (16) were used to compute $E_n(x)$ and $G_n(x)$, and a few values of $D_n(x)$ and $G_n(x)$ for x close to zero were computed ab initio from (14) and (15). In regions where differences are not given, the entries were checked from the Wronskian relations:

$$x[C_1(x)D_0(x)-D_1(x)C_0(x)]=\frac{1}{\pi}, \quad x[E_1(x)G_0(x)+G_1(x)E_0(x)]=\tfrac{1}{2}.$$

All other entries were differenced as functions of x.

In some regions, the number of significant figures given here is the same as in the corresponding entries of [2]; consequently an error of two units in the last place might occasionally go undetected in these regions. This applies to the following sections of the tables:

Table II. $D_0(x)$: all x; $D_1(x)$: $x \leq 2$, $125 \leq x \leq 250$.

Table III. $E_0(x)$: $0 \leq x \leq 1$, $4.05 \leq x \leq 6.2$.

Table IV. $e^{-2\sqrt{x}}E_0(x)$: $6.2 \leq x \leq 100$; $e^{-2\sqrt{x}}E_1(x)$: $65 \leq x \leq 100$.

Table V. $G_0(x)$, $G_1(x)$: $x \leq 1$, $2.5 \leq x \leq 4$.

Table VI. $e^{2\sqrt{x}}G_0(x)$: $x \leq 100$; $e^{2\sqrt{x}}G_1(x)$: $40 \leq x \leq 100$.

All other entries are believed to be correct to within a unit in the last place, and for $x > 100$ to within 0.6 units in the last place.

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June 20, 1949.

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$J_1(2\sqrt{x})/\sqrt{x}$: $x = 0(.02)1.5(.05)3(.1)13(.2)45(.5)115(1)125$, 8D; $125(1)410$, 9D

TABLE I. $J_0(2\sqrt{x})$ and $J_1(2\sqrt{x})/\sqrt{x}$

x	$J_0(2\sqrt{x})$	δ^2	$J_1(2\sqrt{x})/\sqrt{x}$	δ^2
0.00	1.00000 000	20 000	1.00000 000	6 667
0.02	0.98009 978	19 867	0.99003 328	6 633
0.04	0.96039 823	19 735	0.98013 289	6 600
0.06	0.94089 402	19 603	0.97029 850	6 567
0.08	0.92158 585	19 472	0.96052 979	6 534
0.10	0.90247 240	19 342	0.95082 642	6 502
0.12	0.88355 236	19 212	0.94118 807	6 469
0.14	0.86482 444	19 083	0.93161 441	6 437
0.16	0.84628 735	18 954	0.92210 512	6 404
0.18	0.82793 981	18 827	0.91265 986	6 372
0.20	0.80978 053	18 700	0.90327 833	6 340
0.22	0.79180 825	18 573	0.89396 020	6 308
0.24	0.77402 171	18 447	0.88470 514	6 276
0.26	0.75641 963	18 322	0.87551 285	6 245
0.28	0.73900 077	18 197	0.86638 300	6 213
0.30	0.72176 390	18 073	0.85731 528	6 181
0.32	0.70470 775	17 950	0.84830 938	6 150
0.34	0.68783 111	17 827	0.83936 498	6 119
0.36	0.67113 274	17 705	0.83048 176	6 088
0.38	0.65461 143	17 584	0.82165 943	6 057
0.40	0.63826 596	17 463	0.81289 766	6 026
0.42	0.62209 513	17 343	0.80419 615	5 996
0.44	0.60609 772	17 223	0.79555 460	5 965
0.46	0.59027 254	17 104	0.78697 270	5 934
0.48	0.57461 841	16 986	0.77845 014	5 904
0.50	0.55913 414	16 868	0.76998 662	5 874
0.52	0.54381 856	16 751	0.76158 184	5 844
0.54	0.52867 048	16 634	0.75323 550	5 814
0.56	0.51368 875	16 518	0.74494 730	5 784
0.58	0.49887 220	16 403	0.73671 693	5 754
0.60	0.48421 969	16 288	0.72854 411	5 725
0.62	0.46973 006	16 174	0.72042 854	5 695
0.64	0.45540 217	16 060	0.71236 992	5 666
0.66	0.44123 488	15 947	0.70436 796	5 637
0.68	0.42722 707	15 835	0.69642 236	5 608
0.70	0.41337 761	15 723	0.68853 284	5 579
0.72	0.39968 539	15 612	0.68069 911	5 550
0.74	0.38614 928	15 501	0.67292 087	5 521
0.76	0.37276 818	15 391	0.66519 785	5 492
0.78	0.35954 100	15 281	0.65752 974	5 464
0.80	0.34646 663	15 172	0.64991 628	5 435
0.82	0.33354 399	15 064	0.64235 717	5 407
0.84	0.32077 198	14 956	0.63485 213	5 379
0.86	0.30814 954	14 849	0.62740 088	5 351
0.88	0.29567 559	14 742	0.62000 313	5 323
0.90	0.28334 906	14 636	0.61265 862	5 295
0.92	0.27116 889	14 530	0.60536 706	5 267
0.94	0.25913 403	14 425	0.59812 818	5 240
0.96	0.24724 342	14 321	0.59094 169	5 212
0.98	0.23549 601	14 217	0.58380 732	5 185
1.00	0.22389 078	14 113	0.57672 481	5 158

TABLE I. $J_0(2\sqrt{x})$ and $J_1(2\sqrt{x})/\sqrt{x}$

x	$J_0(2\sqrt{x})$	δ^{2*}	$J_1(2\sqrt{x})/\sqrt{x}$	δ^{2*}
1.00	0.22389 078	14 113	0.57672 481	5 158
1.02	0.21242 668	14 010	0.56969 387	5 131
1.04	0.20110 268	13 908	0.56271 424	5 104
1.06	0.18991 777	13 806	0.55578 564	5 077
1.08	0.17887 092	13 705	0.54890 781	5 050
1.10	0.16796 112	13 604	0.54208 048	5 023
1.12	0.15718 736	13 504	0.53530 339	4 997
1.14	0.14654 865	13 404	0.52857 625	4 970
1.16	0.13604 398	13 305	0.52189 882	4 944
1.18	0.12567 237	13 207	0.51527 083	4 918
1.20	0.11543 282	13 109	0.50869 201	4 891
1.22	0.10532 436	13 011	0.50216 210	4 865
1.24	0.09534 601	12 914	0.49568 085	4 839
1.26	0.08549 681	12 817	0.48924 799	4 814
1.28	0.07577 577	12 721	0.48286 326	4 788
1.30	0.06618 196	12 626	0.47652 642	4 762
1.32	0.05671 440	12 531	0.47023 719	4 737
1.34	0.04737 215	12 436	0.46399 534	4 711
1.36	0.03815 427	12 342	0.45780 060	4 686
1.38	0.02905 982	12 249	0.45165 271	4 661
1.40	0.02008 785	12 156	0.44555 144	4 636
1.42	0.01123 745	12 064	0.43949 653	4 611
1.44	+0.00250 768	11 972	0.43348 772	4 586
1.46	-0.00610 237	11 880	0.42752 478	4 561
1.48	-0.01459 361	11 789	0.42160 745	4 537
1.50	-0.02296 697	11 699	0.41573 548	4 512
1.50	-0.02296 697	73 115	0.41573 548	28 200
1.55	-0.04339 050	71 715	0.40125 243	27 819
1.60	-0.06309 685	70 333	0.38704 758	27 442
1.65	-0.08209 983	68 970	0.37311 715	27 069
1.70	-0.10041 307	67 626	0.35945 743	26 701
1.75	-0.11805 002	66 3[illegible]0	0.34606 472	26 336
1.80	-0.13502 393	64 993	0.33293 538	25 975
1.85	-0.15134 789	63 703	0.32006 579	25 617
1.90	-0.16703 478	62 431	0.30745 237	25 264
1.95	-0.18209 734	61 176	0.29509 161	24 914
2.00	-0.19654 810	59 939	0.28297 999	24 568
2.05	-0.21039 943	58 719	0.27111 406	24 226
2.10	-0.22366 354	57 517	0.25949 039	23 887
2.15	-0.23635 245	56 331	0.24810 560	23 552
2.20	-0.24847 802	55 161	0.23695 635	23 221
2.25	-0.26005 195	54 008	0.22603 931	22 893
2.30	-0.27108 577	52 872	0.21535 120	22 569
2.35	-0.28159 084	51 752	0.20488 880	22 248
2.40	-0.29157 836	50 647	0.19464 888	21 931
2.45	-0.30105 938	49 558	0.18462 828	21 617
2.50	-0.31004 479	48 485	0.17482 385	21 307

Table I. $J_0(2\sqrt{x})$ and $J_1(2\sqrt{x})/\sqrt{x}$

x	$J_0(2\sqrt{x})$	δ^{2*}	$J_1(2\sqrt{x})/\sqrt{x}$	δ^{2*}
2.50	-0.31004 479	48 485	0.17482 385	21 307
2.55	-0.31854 532	47 428	0.16523 250	21 000
2.60	-0.32657 154	46 385	0.15585 116	20 696
2.65	-0.33413 388	45 358	0.14667 678	20 396
2.70	-0.34124 262	44 346	0.13770 636	20 099
2.75	-0.34790 787	43 348	0.12893 694	19 805
2.80	-0.35413 961	42 365	0.12036 558	19 515
2.85	-0.35994 768	41 397	0.11198 936	19 227
2.90	-0.36534 175	40 442	0.10380 543	18 943
2.95	-0.37033 138	39 502	0.09581 093	18 662
3.00	-0.37492 596	38 576	0.08800 306	18 384
3.00	-0.37492 596	154 287	0.08800 306	73 534
3.10	-0.38296 688	147 044	0.07293 613	71 348
3.20	-0.38953 697	140 016	0.05858 276	69 211
3.30	-0.39470 650	133 200	0.04492 160	67 122
3.40	-0.39854 366	126 590	0.03193 175	65 080
3.50	-0.40111 455	120 182	0.01959 279	63 084
3.60	-0.40248 325	113 972	+ 0.00788 475	61 134
3.70	-0.40271 188	107 954	- 0.00321 186	59 227
3.80	-0.40186 062	102 125	- 0.01371 612	57 365
3.90	-0.39998 778	96 480	- 0.02364 666	55 545
4.00	-0.39714 981	91 014	- 0.03302 166	53 767
4.10	-0.39340 137	85 725	- 0.04185 892	52 031
4.20	-0.38879 537	80 607	- 0.05017 580	50 335
4.30	-0.38338 299	75 657	- 0.05798 925	48 679
4.40	-0.37721 375	70 870	- 0.06531 585	47 061
4.50	-0.37033 551	66 243	- 0.07217 175	45 483
4.60	-0.36279 456	61 772	- 0.07857 277	43 941
4.70	-0.35463 561	57 454	- 0.08453 430	42 437
4.80	-0.34590 185	53 283	- 0.09007 139	40 969
4.90	-0.33663 499	49 258	- 0.09519 874	39 536
5.00	-0.32687 528	45 375	- 0.09993 065	38 138
5.10	-0.31666 157	41 630	- 0.10428 112	36 775
5.20	-0.30603 132	38 019	- 0.10826 379	35 444
5.30	-0.29502 063	34 540	- 0.11189 195	34 147
5.40	-0.28366 431	31 188	- 0.11517 857	32 882
5.50	-0.27199 588	27 962	- 0.11813 632	31 649
5.60	-0.26004 760	24 858	- 0.12077 753	30 446
5.70	-0.24785 052	21 872	- 0.12311 421	29 274
5.80	-0.23543 452	19 002	- 0.12515 810	28 132
5.90	-0.22282 828	16 244	- 0.12692 060	27 019
6.00	-0.21005 940	13 597	- 0.12841 287	25 935
6.10	-0.19715 436	11 057	- 0.12964 572	24 879
6.20	-0.18413 855	8 620	- 0.13062 974	23 851
6.30	-0.17103 636	6 285	- 0.13137 520	22 849
6.40	-0.15787 113	4 050	- 0.13189 212	21 874
6.50	-0.14466 523	1 910	- 0.13219 025	20 925

TABLE I. $J_0(2\sqrt{x})$ and $J_1(2\sqrt{x})/\sqrt{x}$

x	$J_0(2\sqrt{x})$	δ^{2*}	$J_1(2\sqrt{x})/\sqrt{x}$	δ^{2*}
6.5	-0.14466 523	+ 1 910	- 0.13219 025	20 925
6.6	-0.13144 006	- 136	- 0.13227 908	20 001
6.7	-0.11821 608	- 2 091	- 0.13216 786	19 103
6.8	-0.10501 285	- 3 958	- 0.13186 557	18 228
6.9	-0.09184 904	- 5 738	- 0.13138 095	17 378
7.0	-0.07874 246	- 7 434	- 0.13072 251	16 551
7.1	-0.06571 006	- 9 049	- 0.12989 852	15 747
7.2	-0.05276 800	- 10 584	- 0.12891 703	14 965
7.3	-0.03993 164	- 12 042	- 0.12778 584	14 206
7.4	-0.02721 557	- 13 426	- 0.12651 256	13 468
7.5	-0.01463 362	- 14 737	- 0.12510 456	12 751
7.6	-0.00219 891	- 15 977	- 0.12356 902	12 054
7.7	+0.01007 616	- 17 148	- 0.12191 290	11 379
7.8	0.02217 988	- 18 253	- 0.12014 296	10 722
7.9	0.03410 118	- 19 293	- 0.11826 575	10 086
8.0	0.04582 966	- 20 271	- 0.11628 766	9 468
8.1	0.05735 555	- 21 188	- 0.11421 485	8 869
8.2	0.06866 968	- 22 045	- 0.11205 332	8 288
8.3	0.07976 345	- 22 846	- 0.10980 888	7 725
8.4	0.09062 888	- 23 591	- 0.10748 716	7 179
8.5	0.10125 849	- 24 282	- 0.10509 362	6 650
8.6	0.11164 538	- 24 921	- 0.10263 355	6 138
8.7	0.12178 315	- 25 510	- 0.10011 207	5 642
8.8	0.13166 591	- 26 050	- 0.09753 413	5 162
8.9	0.14128 826	- 26 543	- 0.09490 455	4 698
9.0	0.15064 526	- 26 990	- 0.09222 795	4 249
9.1	0.15973 243	- 27 394	- 0.08950 884	3 815
9.2	0.16854 575	- 27 754	- 0.08675 155	3 396
9.3	0.17708 161	- 28 073	- 0.08396 027	2 990
9.4	0.18533 681	- 28 352	- 0.08113 907	2 599
9.5	0.19330 856	- 28 593	- 0.07829 185	2 221
9.6	0.20099 444	- 28 797	- 0.07542 240	1 857
9.7	0.20839 242	- 28 965	- 0.07253 436	1 505
9.8	0.21550 081	- 29 098	- 0.06963 125	1 166
9.9	0.22231 828	- 29 199	- 0.06671 645	840
10.0	0.22884 382	- 29 267	- 0.06379 323	525
10.1	0.23507 675	- 29 304	- 0.06086 474	+ 223
10.2	0.24101 669	- 29 312	- 0.05793 399	- 68
10.3	0.24666 357	- 29 291	- 0.05500 391	- 348
10.4	0.25201 759	- 29 243	- 0.05207 729	- 617
10.5	0.25707 923	- 29 168	- 0.04915 681	- 875
10.6	0.26184 924	- 29 068	- 0.04624 506	- 1 122
10.7	0.26632 862	- 28 944	- 0.04334 452	- 1 360
10.8	0.27051 860	- 28 796	- 0.04045 755	- 1 587
10.9	0.27442 066	- 28 627	- 0.03758 644	- 1 805
11.0	0.27803 649	- 28 436	- 0.03473 336	- 2 013
11.1	0.28136 800	- 28 224	- 0.03190 040	- 2 212
11.2	0.28441 730	- 27 994	- 0.02908 954	- 2 402
11.3	0.28718 671	- 27 744	- 0.02630 268	- 2 583
11.4	0.28967 870	- 27 477	- 0.02354 164	- 2 756
11.5	0.29189 595	- 27 193	- 0.02080 815	- 2 920

x	$J_0(2\sqrt{x})$	δ^{2*}	$J_1(2\sqrt{x})/\sqrt{x}$	δ^{2*}
11.5	0.29189 595	- 27 193	- 0.02080 815	- 2 920
11.6	0.29384 130	- 26 893	- 0.01810 384	- 3 077
11.7	0.29551 775	- 26 578	- 0.01543 029	- 3 225
11.8	0.29692 844	- 26 249	- 0.01278 897	- 3 366
11.9	0.29807 666	- 25 905	- 0.01018 130	- 3 499
12.0	0.29896 586	- 25 549	- 0.00760 860	- 3 625
12.1	0.29959 959	- 25 181	- 0.00507 213	- 3 743
12.2	0.29998 154	- 24 801	- 0.00257 309	- 3 855
12.3	0.30011 549	- 24 410	- 0.00011 258	- 3 960
12.4	0.30000 537	- 24 009	+ 0.00230 833	- 4 059
12.5	0.29965 518	- 23 598	0.00468 866	- 4 151
12.6	0.29906 902	- 23 179	0.00702 750	- 4 237
12.7	0.29825 109	- 22 751	0.00932 397	- 4 317
12.8	0.29720 567	- 22 315	0.01157 727	- 4 392
12.9	0.29593 710	- 21 873	0.01378 667	- 4 460
13.0	0.29444 982	- 21 423	0.01595 148	- 4 523
13.0	0.29444 982	- 85 702	0.01595 148	-18 100
13.2	0.29083 713	- 82 036	0.02014 484	-18 541
13.4	0.28640 423	- 78 290	0.02415 294	-18 902
13.6	0.28118 855	- 74 480	0.02797 216	-19 187
13.8	0.27522 815	- 70 621	0.03159 965	-19 400
14.0	0.26856 162	- 66 725	0.03503 326	-19 548
14.2	0.26122 788	- 62 806	0.03827 150	-19 632
14.4	0.25326 610	- 58 876	0.04131 353	-19 659
14.6	0.24471 556	- 54 946	0.04415 908	-19 630
14.8	0.23561 555	- 51 027	0.04680 841	-19 551
15.0	0.22600 523	- 47 128	0.04926 232	-19 425
15.2	0.21592 357	- 43 260	0.05152 205	-19 255
15.4	0.20540 924	- 39 429	0.05358 931	-19 044
15.6	0.19450 054	- 35 644	0.05546 619	-18 796
15.8	0.18323 529	- 31 913	0.05715 518	-18 513
16.0	0.17165 081	- 28 241	0.05865 909	-18 199
16.2	0.15978 379	- 24 635	0.05998 107	-17 855
16.4	0.14767 028	- 21 101	0.06112 454	-17 485
16.6	0.13534 563	- 17 643	0.06209 321	-17 091
16.8	0.12284 439	- 14 266	0.06289 101	-16 675
17.0	0.11020 034	- 10 974	0.06352 209	-16 240
17.2	0.09744 638	- 7 771	0.06399 080	-15 788
17.4	0.08461 454	- 4 660	0.06430 166	-15 320
17.6	0.07173 593	- 1 644	0.06445 934	-14 839
17.8	0.05884 069	+ 1 275	0.06446 866	-14 346
18.0	0.04595 803	4 094	0.06433 453	-13 844
18.2	0.03311 611	6 812	0.06406 198	-13 333
18.4	0.02034 212	9 426	0.06365 612	-12 815
18.6	+0.00766 221	11 937	0.06312 211	-12 293
18.8	-0.00489 853	14 343	0.06246 518	-11 766
19.0	-0.01731 602	16 644	0.06169 060	-11 237

Table I. $J_0(2\sqrt{x})$ and $J_1(2\sqrt{x})/\sqrt{x}$

x	$J_0(2\sqrt{x})$	δ^{2*}	$J_1(2\sqrt{x})/\sqrt{x}$	δ^{2*}
19.0	-0.01731 602	16 644	0.06169 060	-11 237
19.2	-0.02956 728	18 838	0.06080 365	-10 706
19.4	-0.04163 034	20 926	0.05980 964	-10 176
19.6	-0.05348 435	22 908	0.05871 387	- 9 646
19.8	-0.06510 946	24 785	0.05752 164	- 9 118
20.0	-0.07648 693	26 556	0.05623 823	- 8 593
20.2	-0.08759 903	28 222	0.05486 888	- 8 071
20.4	-0.09842 910	29 784	0.05341 881	- 7 555
20.6	-0.10896 151	31 244	0.05189 319	- 7 043
20.8	-0.11918 167	32 602	0.05029 712	- 6 538
21.0	-0.12907 600	33 860	0.04863 565	- 6 039
21.2	-0.13863 191	35 018	0.04691 378	- 5 548
21.4	-0.14783 781	36 080	0.04513 641	- 5 065
21.6	-0.15668 310	37 045	0.04330 838	- 4 590
21.8	-0.16515 811	37 916	0.04143 444	- 4 124
22.0	-0.17325 412	38 695	0.03951 923	- 3 667
22.2	-0.18096 335	39 384	0.03756 734	- 3 221
22.4	-0.18827 891	39 984	0.03558 321	- 2 784
22.6	-0.19519 478	40 498	0.03357 124	- 2 358
22.8	-0.20170 583	40 928	0.03153 566	- 1 942
23.0	-0.20780 775	41 276	0.02948 065	- 1 537
23.2	-0.21349 706	41 544	0.02741 024	- 1 144
23.4	-0.21877 108	41 734	0.02532 837	- 762
23.6	-0.22362 790	41 849	0.02323 885	- 392
23.8	-0.22806 636	41 891	0.02114 540	- 33
24.0	-0.23208 603	41 863	0.01905 159	+ 314
24.2	-0.23568 720	41 767	0.01696 091	649
24.4	-0.23887 083	41 604	0.01487 669	972
24.6	-0.24163 853	41 379	0.01280 217	1 283
24.8	-0.24399 255	41 092	0.01074 046	1 582
25.0	-0.24593 576	40 747	0.00869 455	1 869
25.2	-0.24747 162	40 345	0.00666 731	2 144
25.4	-0.24860 412	39 890	0.00466 150	2 408
25.6	-0.24933 782	39 383	0.00267 973	2 659
25.8	-0.24967 778	38 827	+ 0.00072 454	2 898
26.0	-0.24962 956	38 224	- 0.00120 169	3 126
26.2	-0.24919 919	37 577	- 0.00309 669	3 342
26.4	-0.24839 312	36 888	- 0.00495 828	3 547
26.6	-0.24721 824	36 159	- 0.00678 443	3 740
26.8	-0.24568 183	35 393	- 0.00857 320	3 922
27.0	-0.24379 157	34 591	- 0.01032 278	4 093
27.2	-0.24155 545	33 757	- 0.01203 145	4 252
27.4	-0.23898 182	32 891	- 0.01369 762	4 401
27.6	-0.23607 934	31 997	- 0.01531 979	4 540
27.8	-0.23285 693	31 076	- 0.01689 659	4 668
28.0	-0.22932 381	30 130	- 0.01842 673	4 785
28.2	-0.22548 943	29 162	- 0.01990 903	4 893
28.4	-0.22136 346	28 174	- 0.02134 242	4 991
28.6	-0.21695 579	27 167	- 0.02272 592	5 079
28.8	-0.21227 648	26 143	- 0.02405 864	5 158
29.0	-0.20733 577	25 104	- 0.02533 980	5 227

TABLE I. $J_0(2\sqrt{x})$ and $J_1(2\sqrt{x})/\sqrt{x}$

x	$J_0(2\sqrt{x})$	δ^{2*}	$J_1(2\sqrt{x})/\sqrt{x}$	δ^{2*}
29.0	-0.20733 577	25 104	- 0.02533 980	5 227
29.2	-0.20214 404	24 053	- 0.02656 871	5 288
29.4	-0.19671 181	22 990	- 0.02774 476	5 340
29.6	-0.19104 970	21 917	- 0.02886 743	5 383
29.8	-0.18516 843	20 837	- 0.02993 628	5 418
30.0	-0.17907 880	19 751	- 0.03095 098	5 444
30.2	-0.17279 167	18 660	- 0.03191 124	5 463
30.4	-0.16631 794	17 566	- 0.03281 689	5 474
30.6	-0.15966 856	16 471	- 0.03366 781	5 478
30.8	-0.15285 447	15 375	- 0.03446 396	5 475
31.0	-0.14588 662	14 281	- 0.03520 537	5 465
31.2	-0.13877 596	13 190	- 0.03589 215	5 448
31.4	-0.13153 339	12 102	- 0.03652 446	5 424
31.6	-0.12416 979	11 021	- 0.03710 254	5 395
31.8	-0.11669 597	9 945	- 0.03762 669	5 359
32.0	-0.10912 269	8 877	- 0.03809 725	5 318
32.2	-0.10146 061	7 818	- 0.03851 465	5 271
32.4	-0.09372 034	6 769	- 0.03887 936	5 218
32.6	-0.08591 235	5 731	- 0.03919 189	5 161
32.8	-0.07804 702	4 705	- 0.03945 281	5 099
33.0	-0.07013 462	3 692	- 0.03966 276	5 032
33.2	-0.06218 527	2 693	- 0.03982 240	4 961
33.4	-0.05420 897	1 708	- 0.03993 244	4 885
33.6	-0.04621 555	+ 739	- 0.03999 363	4 806
33.8	-0.03821 472	- 214	- 0.04000 678	4 722
34.0	-0.03021 599	- 1 150	- 0.03997 271	4 635
34.2	-0.02222 872	- 2 068	- 0.03989 229	4 545
34.4	-0.01426 210	- 2 968	- 0.03976 643	4 452
34.6	-0.00632 512	- 3 848	- 0.03959 605	4 356
34.8	+0.00157 341	- 4 710	- 0.03938 212	4 256
35.0	0.00942 489	- 5 551	- 0.03912 563	4 155
35.2	0.01722 089	- 6 371	- 0.03882 760	4 051
35.4	0.02495 322	- 7 171	- 0.03848 906	3 944
35.6	0.03261 389	- 7 949	- 0.03811 109	3 836
35.8	0.04019 510	- 8 705	- 0.03769 476	3 726
36.0	0.04768 931	- 9 439	- 0.03724 118	3 614
36.2	0.05508 917	- 10 150	- 0.03675 147	3 500
36.4	0.06238 756	- 10 839	- 0.03622 675	3 386
36.6	0.06957 761	- 11 505	- 0.03566 818	3 270
36.8	0.07665 266	- 12 147	- 0.03507 691	3 153
37.0	0.08360 628	- 12 766	- 0.03445 412	3 035
37.2	0.09043 228	- 13 361	- 0.03380 098	2 916
37.4	0.09712 472	- 13 932	- 0.03311 867	2 797
37.6	0.10367 788	- 14 480	- 0.03240 840	2 678
37.8	0.11008 630	- 15 003	- 0.03167 135	2 558
38.0	0.11634 472	- 15 503	- 0.03090 872	2 438
38.2	0.12244 816	- 15 978	- 0.03012 171	2 318
38.4	0.12839 186	- 16 430	- 0.02931 153	2 198
38.6	0.13417 130	- 16 857	- 0.02847 937	2 078
38.8	0.13978 222	- 17 261	- 0.02762 643	1 958
39.0	0.14522 057	- 17 641	- 0.02675 390	1 839