The Finite Element Method in Heat Transfer Analysis

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Preface

Over the past three decades, the use of numerical simulation with high-speed electronic computers has gained wide acceptance throughout most of the major branches of engineering. Our joint research work on the subject of the use of the finite element method in heat transfer analysis began in the mid 1970s and flourished in such a way that we decided to write this text.

The text has evolved in much the same way that our joint research did. First, we went through steady-state heat conduction problems, eventually increasing in complexity to include such difficulties as the non-linear analysis of coupled heat and mass transfer in capillary porous bodies and the calculation of shrinkage stresses using an elasto-plastic constitutive relationship. Our lectures at post-graduate level at our institutions and at finite element courses in different parts of the world have also been a stimulus to complete the text.

The need for the book stems from the fact that few texts of this kind exist despite a plethora of classical work on heat transfer problems. The topic is itself briefly covered in many texts on the finite element method, but the detail covered in this book has been available only in research papers up until now. There has been literally an explosion of research interest in the area and it has been impossible to reference every work since the early 1970s, as a spectrum of disciplines is involved.

The arrangement of the text proceeds in a logical order of complexity. The first chapter deals with the importance of heat transfer in engineering problems and derives the general heat conduction equation. Then the weak variational formulation and appropriate initial, boundary and interfacial conditions are discussed. In the second chapter, linear steady-state heat conduction problems are solved using the Galerkin form of the weak formulation. Initially the focus is on problems involving one space dimension in an attempt to develop concepts and to aid understanding of the process of matrix assembly. Solutions to two-dimensional problems are then given using rectangular elements. The solutions are worked for single and multiple elements to bring out the importance of the number of elements in the solution. Chapter 3 deals with the various time stepping schemes used in unsteady state problems, including topics such as stability and convergence of solutions. Chapter 4 deals with the solution of transient non-linear heat conduction problems and provides a number of examples. The problems of melting and solidification are dealt with at length in Chapter 5 in view of their practical importance in manufacturing processes like casting and welding, for example. The difficulties associated with the application of the finite element method to convection problems and the methods used to overcome them are given in Chapter 6, along with simple illustrative examples of convective heat transfer between two parallel heated plates.

In order to bring out the practical importance of heat transfer analysis in the engineering industry, Chapter 7 deals with the application of heat and mass transfer to drying problems and the calculation of both thermal and shrinkage stresses.

The preface would not be complete without acknowledging the continued support that we have received over the years from our respective research councils, industrial partners, The British Council, and various other funding sources, together with the invaluable assistance given by a large number of talented research associates, assistants and students with whom we have had the pleasure of working.

February 1995

Contents

1 Conduction Heat Transfer and Formulation

1.1 INTRODUCTION

There are many practical engineering problems that require the analysis of problems involving the transfer of heat. The solution of the equation of heat conduction is sufficient in many cases, while the area of application expands considerably if the equations governing coupled heat and mass transfer and/or thermal convection are considered. Combining a thermal analysis procedure with a thermal stress predictive capability can provide answers to questions of immediate concern in many industrial processes.

In the field of civil engineering, there are numerous examples of practical problems where the behaviour of the system under consideration may be predicted via a heat transfer analysis. The thermal properties of materials used for construction purposes often necessitate the use of some form of insulation to counter the effects of the wide range of temperatures produced by seasonal climatic changes. This is particularly true in highway construction, where it is important to predict the placement and quantity of insulating material in order to prevent possible damage to the road surface due to frost heave. At excavation sites, where the soil may be unstable because of friability or water saturation, artificial freezing of the soil is often employed to render it structurally stable. This typically involves the circulation of a refrigerated liquid through pipes sunk in concentric circular patterns around the excavation site. The high cost of the process, in terms of the refrigeration plant and energy consumption, means that a numerical simulation of the process can prove to be extremely useful. If the numerical method makes accurate predictions of the advance of the frozen region, the numerical results can be used to aid the design of the optimal arrangement of the cooling pipes and to predict the required running time necessary to produce an ice wall of the required thickness. In the foundry, it is imperative to understand the process of solidification in metal casting and the effect this produces on porosity formation. In high pressure die casting, it is often required to design a cooling system that will ensure the desired surface finish. Problems of this type can be investigated by a heat conduction analysis, taking account of latent heat effects and allowing for the variation in the thermal properties of the materials with temperature.

For the simulation of problems involving porous capillary materials, such as timber or ceramics, a coupled heat and mass transfer analysis is required. The optimal drying rate to ensure a reasonably stress-free end product will be tied in with the energy cost of the kiln schedule. Mathematical models, which can accurately predict the movement of moisture and heat in porous materials, are often the only means of gaining a better insight into the physical process. Similar models prove useful in designing geothermal energy extraction systems or in the analysis of advanced oil recovery by thermal methods. By increasing the complexity of the governing equations, it is possible to predict the aerodynamic heating of structures, such as re-entry vehicles, or of turbine blades in a jet engine. A related problem here is the computation of the thermally induced stresses and the prediction of the working life of the component. Such information is essential in the safety analysis of nuclear reactors and also in the modern steel industry, where the production rate of continuously cast steel is limited by the cracking induced by thermal stress development in the strand.

Exact analytical solutions of the governing equations of heat transfer can only be obtained for problems in which restrictive simplifying assumptions have been made with respect to geometry, material properties and boundary conditions. There is therefore no option but to turn to numerical solution methods for the analysis of practical problems, where such simplifications are not generally possible. The finite element method, with its flexibility in dealing with complex geometries, is an ideal approach to employ in the solution of such problems. However, the newcomer frequently finds the gap between finite element theory and practice quite daunting and it is the objective of this book to attempt to bridge this gap. The material included proceeds in an orderly fashion from the governing differential equations to the finite element formulation.

Initially, the basic differential equation governing heat conduction will be derived and the concept of a variational formulation of the problem will be introduced. The finite element method will then be used to generate a system of simultaneous equations which have to be solved to obtain the approximation to the temperature field in the body of interest. The process will be demonstrated in detail for the simple case of linear, steady-state problems in both one and two dimensions. For the solution of time-dependent (transient) problems, finite difference methods will be employed to determine the variation of the solution with time. Different time marching algorithms are presented and their relative merits discussed.

When the variation of the thermal properties of the materials are included in the mathematical model, the resulting equations are non-linear and techniques for dealing with this added complication are described. Some examples for which exact solutions are available are also given for one-, two- and three-dimensional problems. Solution of the non-linear problems can be extended to include the effects of latent heat addition or removal as in melting and solidification problems. These types of problems, along with some practical examples, are dealt with in the chapter on phase change. Convection heat transfer is important when the heat exchange is between a solid and a fluid. The solution methods for such problems and the difficulties encountered in the application of the standard Galerkin method are also dealt with. As has been noted above, there are many practical problems where the determination of the temperature (or moisture) distribution is only required as a prerequisite for the calculation of the resulting stress field. Methods for the determination of thermal or drying stresses in such problems will be detailed.

1.2 MODELLING OF HEAT CONDUCTION

1.2.1 Derivation of the governing equation

The equation governing the conduction of heat in a continuous medium can be derived by imposing the principle of conservation of heat energy over an arbitrary fixed volume, V, of the medium which is bounded by a closed surface S. For convenience the conservation statement is expressed in rate form and is written as:

$$\begin{aligned}\text{rate of increase of heat in } V = {} & \text{rate of heat conduction into } V \text{ across } S \\ & + \text{rate of heat generation within } V\end{aligned} \tag{1.2.1}$$

If u denotes the specific internal energy of the medium, then

$$\text{rate of increase of heat in } V = \int_V \rho \frac{\partial u}{\partial t} \,\mathrm{d}V \tag{1.2.2}$$

where ρ is the density of the medium. Introducing the specific heat, c, of the medium defined by

$$c = \frac{\mathrm{d}u}{\mathrm{d}T} \tag{1.2.3}$$

where T is the temperature, means that we can write equation (1.2.2) as

$$\text{rate of increase of heat in } V = \int_V \rho c \frac{\partial T}{\partial t} \,\mathrm{d}V \tag{1.2.4}$$

To obtain an expression for the rate at which heat is conducted into V across S, we make use of Fourier's Law of Conduction. This is an empirical relationship which states that, for a surface with unit normal vector $\mathbf{n}$, the rate at which heat is conducted across the surface, per unit area, in the direction of $\mathbf{n}$ is given by

$$q = -k(\text{grad } T)\cdot\mathbf{n} = -k\frac{\partial T}{\partial n} \tag{1.2.5}$$

where k is a property of the medium termed the thermal conductivity. In this equation $\partial/\partial n$ denotes differentiation in the direction of $\mathbf{n}$ and q is termed the flux of heat in this direction. Thus, if $\mathbf{n}$ denotes the outward unit normal to S, it follows that

$$\text{rate of heat conduction into } V \text{ across } S$$
$$= \int_S -q\,\mathrm{d}S = \int_S k(\operatorname{grad} T)\cdot\mathbf{n}\,\mathrm{d}S = \int_V \operatorname{div}(k \operatorname{grad} T)\,\mathrm{d}V \qquad (1.2.6)$$

where the Divergence Theorem has been applied. If it is assumed that heat generation in the medium is occurring at a rate Q per unit volume, then

$$\text{rate of heat generation within } V = \int_V Q\,\mathrm{d}V \qquad (1.2.7)$$

Using equations (1.2.4), (1.2.6) and (1.2.7) in (1.2.1) produces the conservation statement:

$$\int_V \left(\rho c \frac{\partial T}{\partial t} - \operatorname{div}(k \operatorname{grad} T) - Q\right)\mathrm{d}V = 0 \qquad (1.2.8)$$

and, since the volume V was arbitrarily chosen initially, it follows that

$$\rho c \frac{\partial T}{\partial t} = \operatorname{div}(k \operatorname{grad} T) + Q \qquad (1.2.9)$$

everywhere in the medium. This is the familiar form of the heat conduction equation for a non-stationary system.

If the medium is anisotropic, i.e. the conductivity depends upon the direction, the form of the heat conduction equation is modified to

$$\rho c \frac{\partial T}{\partial t} = \operatorname{div}(\mathbf{k} \operatorname{grad} T) + Q \qquad (1.2.10)$$

where

$$\mathbf{k} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \qquad (1.2.11)$$

is a conductivity tensor and, for example, k_{xy} denotes the thermal conductivity in the x direction across a surface with normal in the y direction. If the conductivity k and the specific heat capacity ρc are assumed to be constant, and if the heat generation rate Q is independent of T, then equation (1.2.9) is linear and can be written as

$$\frac{1}{\alpha_1}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{Q}{k} \qquad (1.2.12)$$

where $\alpha_1 = k/\rho c$ is termed the thermal diffusivity of the medium and ∇^2 denotes the Laplacian operator defined, in Cartesian coordinates, by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.2.13}$$

In the absence of heat generation within the medium, equation (1.2.12) reduces to the standard diffusion equation:

$$\frac{1}{\alpha_1}\frac{\partial T}{\partial t} = \nabla^2 T \tag{1.2.14}$$

If, in addition, the temperature does not vary with time, steady-state conditions are said to exist and, in this case, the governing equation simplifies further to

$$\nabla^2 T = 0 \tag{1.2.15}$$

which is just the Laplace equation.

1.2.2 Initial and boundary conditions

Suppose that the solution of the heat conduction equation (1.2.9) is required over an arbitrary domain Ω bounded by a closed surface, Γ, as illustrated in Figure 1.2.1. If the problem being modelled is independent of time (i.e. steady), the solution will be uniquely defined provided that we are able to supply appropriate boundary conditions. For the steady heat conduction equation, one condition has to be specified at each point of the boundary curve Γ and typical conditions of practical interest would be:

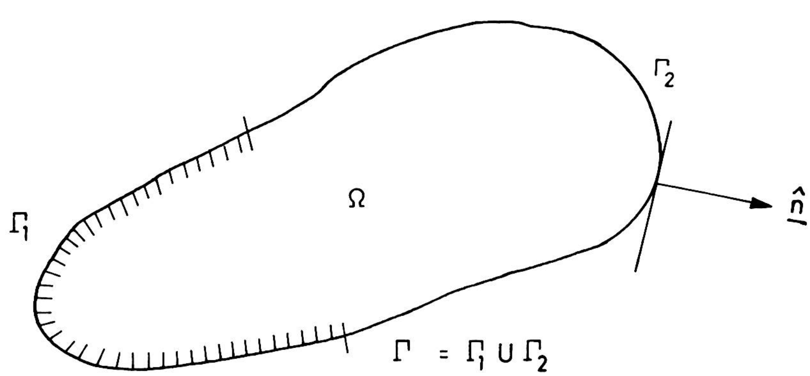

Figure 1.2.1 *General domain and boundary.*

(a) the value of the temperature is prescribed, e.g.

$$T = f(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ on } \Gamma_1 \tag{1.2.16}$$

or

(b) the value of the outward normal heat flux is prescribed, e.g.

$$q = -k(\text{grad } T)\cdot\mathbf{n} = -k\frac{\partial T}{\partial n} = \aleph(\mathbf{x}, T) + \aleph_c(\mathbf{x}, T) + \aleph_r(\mathbf{x}, T) \quad \text{for all } \mathbf{x} \text{ on } \Gamma_2 \tag{1.2.17}$$

Here $f, \aleph, \aleph_c$ and $\aleph_r$ are prescribed functions of $\mathbf{x}$ and T and $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = 0$. In equation (1.2.17) $\aleph$ denotes a specified heat flux; $\aleph_c$ denotes a convective heat flux, defined as

$$\aleph_c = \alpha(T - T_\infty) \tag{1.2.18}$$

where α is a coefficient of surface heat transfer and T_∞ is the specified ambient temperature of the surrounding medium; $\aleph_r$ is a radiative heat flux which is defined as

$$\aleph_r = \varepsilon\sigma(T^4 - T_\infty^4) \tag{1.2.19}$$

where σ is the Stefan–Boltzman constant and ε is the emissivity of the surface, defined as the ratio of the heat emitted by the surface to the heat emitted by a black body at the same temperature.

When the problem being modelled is time dependent (transient), the solution is uniquely determined provided that an initial condition is given together with a boundary condition at each point of the boundary Γ of the domain. The initial condition should give the distribution of the temperature over the entire region Ω at an initial time, usually taken to be the time $t = 0$. In addition, in a transient problem, the functions $f, \aleph, \aleph_c$ and $\aleph_r$ of equations (1.2.16) and (1.2.17) may vary with time.

1.2.3 Interface conditions

When the region Ω, consists of two or more different materials, as shown in Figure 1.2.2, boundary conditions of continuity of temperature and flux across material interfaces have to be applied in order to uniquely define the solution. If we consider an interface $\tilde{\Gamma}$ between two materials, designated Ω_1 and Ω_2, with unit normal vector $\tilde{\mathbf{n}}$ in the direction into Ω_1, these conditions can be written in the form

$$\left.\begin{aligned} T_1(\mathbf{x}, t) &= T_2(\mathbf{x}, t) \\ \tilde{\mathbf{n}}\cdot(k \text{ grad } T)_1 &= \tilde{\mathbf{n}}\cdot(k \text{ grad } T)_2 \end{aligned}\right\} \quad \text{for all } \mathbf{x} \text{ on } \tilde{\Gamma} \text{ and all } t > 0 \tag{1.2.20}$$

where the subscripts 1 and 2 denote the conditions appropriate to regions Ω_1 and Ω_2, respectively.

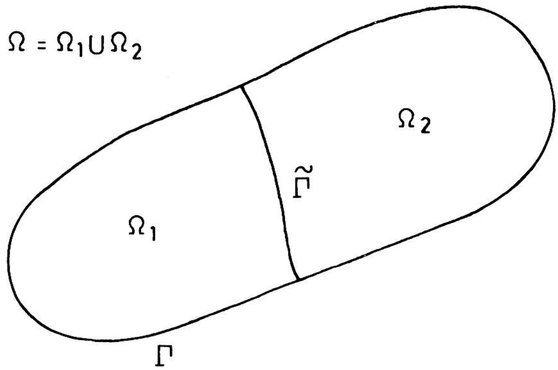

Figure 1.2.2 *Composite material domain and interface.*

1.2.4 Initial, boundary and interface conditions in a practical example

An example of a realistic physical problem is now used to illustrate how the correct boundary and initial conditions may be identified in practice. A tapered slab of aluminium bronze is cast in a resin bonded silica sand mould and a thermal analysis is to be made of the process. This problem involves the added complication that the metal is initially liquid and undergoes a change of phase during the transient. Methods of handling phase change problems will be introduced in Chapter 5 and the complications associated with the phase change process can be ignored for the purposes of this illustration. To reduce computational costs, the decision is made to undertake a two-dimensional analysis, and a vertical cross-section through the mould and casting, shown in Figure 1.2.3, is examined. In the actual problem, metal at 1156 °C is poured into the mould through the riser entrance CD. The simulation of the pouring stage can be expected to pose additional difficult problems and so the conduction analysis will start from the time when the mould is full, with the metal taken to be at a uniform initial temperature of 1156 °C. The external boundary EFGHAB of the sand will be assumed to be maintained at a constant temperature of 20 °C while the initial temperature in the sand is also taken to be at this value. The upper boundary, CD, of the riser will be held constant at a temperature 1156 °C, while the sides BC and DE of the riser will be assumed to be perfectly insulated and therefore subjected to a boundary condition of zero normal heat flux. The initial temperature at the interface between the mould and the metal is taken to be 1080 °C, which is the liquidus temperature of the metal and perfect conduction is assumed at the interface during the transient so that the boundary conditions of equation (1.2.20) may be applied.