B. VAN DER POL

SELECTED SCIENTIFIC PAPERS

BALTHASAR VAN DER POL

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WITH AN INTRODUCTION BY
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VOLUME I

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VOLUME I

BALTHASAR VAN DER POL (1889–1959)

PREFACE

By these two volumes we attempt to cover the total scientific work of Balthasar van der Pol,—in so far as this work is laid down in his published papers. He owned many patents in the field of radio science, and after his death he left a tremendous number of notes (partly stenographic) which may contain still useful material. However, that part of his work as well as other activities as a scientist cannot be included here.

Several of Van der Pol's early papers have appeared in more than one language; where possible, we have reproduced the English version. Surveys and papers of a purely didactical nature have mostly been omitted if a detailed paper was available. Of course, his text-book on Operational Calculus, written jointly with one of us, could not be included either. Van der Pol's doctoral dissertation (propagation of electromagnetic waves in ionized gases, and its application to wireless telegraphy and glow-discharge measurements) appears here in its original form. Readers not familiar with the Dutch language may find compensation in referring to various papers published earlier in English and reproduced here, which contain the main results of the dissertation.

A complete list of Van der Pol's scientific publications will be found at the end of the second volume. The Selected Scientific Papers covers about half of the titles in that list.

We are gratefully indebted to Mrs. P. van der Pol for her advice and help in editing the scientific work of her late husband.

It is a pleasure to thank the publishers and editors of the various journals for their collaboration in permitting the reproduction of the original articles. Financial support of N.V. Philips' Gloeilampenfabrieken is gratefully acknowledged. Finally, our thanks are due to the North-Holland Publishing Company for their cooperation in making this publication possible.

Eindhoven, August 1960

Philips Research Laboratories

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C. J. Bouwkamp

INTRODUCTION

Balthasar van der Pol was a remarkable man in many different ways. He was a prominent scientist and a brilliant lecturer but he will also be remembered as a prudent diplomat and as a wise and impartial chairman of international meetings. His skill in many fields was outstanding: he was a fluent linguist and an erudite musician with a thorough knowledge of harmony and counterpoint and a very sharp sense of absolute pitch; as a stenographer he could vie with a professional. These skills were not separate hobbies; with Van der Pol they rather appeared to be natural and indispensable elements of his creative activity. He was interested in mathematical aspects of music but equally sensitive to the beauty and harmony of mathematical theory; there was even a similarity between his proficiency in shorthand and the virtuosity with which he handled analytical formalism. Those who knew him more closely will gratefully recollect inspiring discussions and playful debates and cherish the memory of a charming and generous host.

Van der Pol was born at Utrecht on the 27th of January 1889. His father, a well-to-do merchant, was a man of broad cultural interests and young Van der Pol had every opportunity to develop his many gifts. The only stumbling-block during his early years were the classics, and he spent several years preparing an examination ("Staatsexamen") required for entering a university, devoting most of his time to music and to chess which he found more interesting. This little episode is characteristic of Van der Pol who also in later life would show an almost contemptuous disregard for things he considered uninteresting or unimportant, a trait that enabled him to concentrate on things he really wanted to do. He was pleased when he discovered that also Oliver Heaviside held the classics in low esteem. For a while Van der Pol hesitated between music, medicine and physics but in 1911 he entered Utrecht University as a student of mathematics and physics. After passing his "doctoraal examen" (roughly equivalent to a master's degree) at Utrecht in 1916, he went to England where he first worked with J. A. Fleming at London and from 1917 to 1919 with J. J. Thomson in the Cavendish Laboratory at Cambridge. His lifelong friendship with E. V. Appleton and many other British scientists dates from those years. In 1919 he returned to the Netherlands and for three years he worked with H. A. Lorentz at "Teyler's Stichting" at Haarlem. Teyler's Foundation is a privately endowed institution, owning a museum, a fine library and a small research laboratory. A special position had been created for Lorentz in order to relieve him of most of his teaching duties at Leyden University, Van der Pol's position being that of a "conservator". In 1920 he obtained his doctor's degree at Utrecht presenting a thesis on the propagation of radio waves in an ionized gas based on experimental work done at Cambridge. In 1922 he joined the research laboratories of the Philips Company at Eindhoven where he stayed until 1949, in later years as "Director of Fundamental Radio Research". From 1938 onwards he was also part-time professor of theoretical electricity at Delft. During all these years Van der Pol did not only keep up an untiring research activity, he also played an important role in the scientific life of his country. He was founder, president and finally honorary member of the "Nederlands Radiogenootschap" (Dutch Institution of Radio Engineers), co-founder of the Dutch journal "Physica", and he became an honorary president of the U.R.S.I.; he was a member of the Royal Dutch Academy of Sciences at Amsterdam, served on many committees and represented the Netherlands at international conferences on wavelength allocation. In 1949, on reaching the age of retirement with the Philips Company, he became director of the C.C.I.R. (Comité Consultatif International des Radiocommunications) at Geneva. In 1956 he laid down this position and retired to Wassenaar (near the Hague). In 1957 he was visiting professor at the University of California at Berkeley, in 1958 at Cornell University at Ithaca. He died on the 6th of October 1959.

Van der Pol's published works, while reflecting the catholicity of his interests, can broadly speaking be classified in three categories: propagation of radio waves, theory of circuits, and general mathematical methods.

His thesis has already been mentioned. Van der Pol himself has not followed up this work and in his later papers little attention is paid to the behaviour of matter at high frequencies. On the other hand the mathematics of the propagation of radio waves was extensively studied by Van der Pol and his co-workers. Together with Bremmer he treated the diffraction around a spherical earth of waves emitted by a dipole; the solution they obtained is not only a mathematical "tour de force" but is also of great practical use.

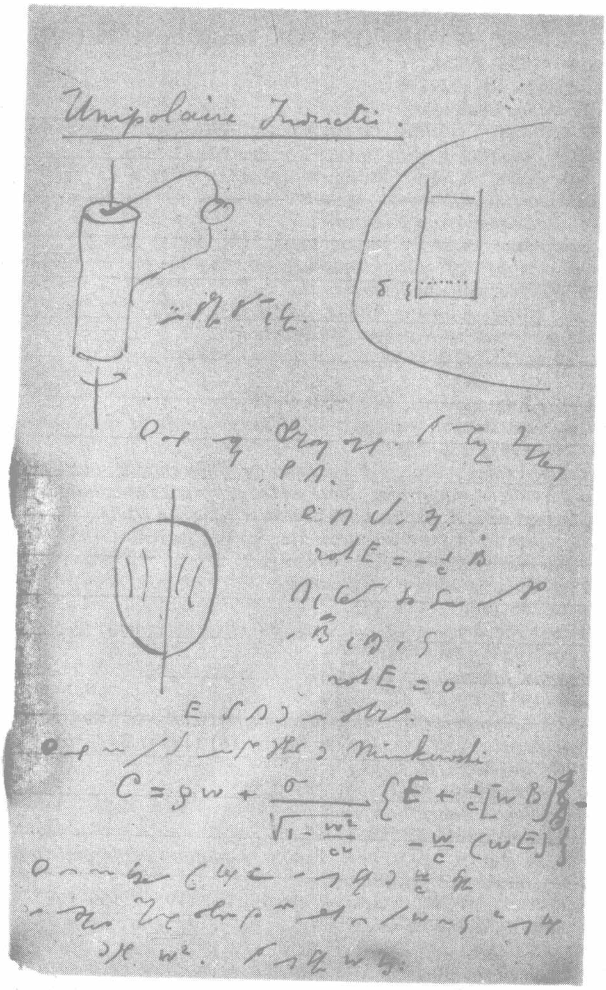
Equally important is his work on non-linear oscillations. Of course periodic oscillations had been thoroughly studied in theoretical mechanics and acoustics but in radio they appeared in a new light and Van der Pol has done much to clarify the subject. After pointing out that the amplitude in an oscillating circuit driven by a triode is determined by non-linear terms in the triode characteristic, he formulated a differential equation that gives an adequate—though somewhat idealized—representation of the actual situation. He showed that there are two limiting cases: the case of nearly linear vibrations and that of "relaxation oscillations". This systematic treatment of such oscillations has greatly helped to elucidate many interesting features especially in connection with synchronization, and "Van der Pol's equation" has become one of the classical equations of advanced circuitry. In this connection the "artificial heart" should also be mentioned. Together with Van der Mark he designed a circuit whose oscillations closely imitate an electrocardiogram. By modification of the circuit parameters the effect of various diseases of the heart can be simulated. This work, besides showing Van der Pol's medical interests, affords an example of a tendency that is present throughout his work: the tendency to look for different phenomena that are described by the same equations. This way of thinking often leads to practical "analogue computers"; it may also throw illuminating sidelights on many problems, the only danger being that one may overrate the importance of a formal analogy and thus fail to see a difference in physical content.

In his earlier papers Van der Pol used mathematics as a tool but later on he became more and more interested in mathematics for its own sake. He was fascinated by Heaviside's operational calculus and the theory of Laplace transformations which are indispensable tools in circuit theory; together with Bremmer he published a substantial volume on a slightly different formulation of Heaviside's method. This book contains a wealth of analytical results and perhaps it may be said that Van der Pol was on the whole more interested in elegant explicit formulae than in general theorems. In later life the theory of numbers became one of his main preoccupations.

The impact of Van der Pol's work can be stated in still another way. Although the study of electromagnetic waves was originally one of the most fundamental branches of physics, radio might have remained a field of haphazard empiricism along with wild commercial ventures, but for the influence of men like Van der Pol who stressed the need for a more scientific approach. Perhaps he has occasionally overemphasized formal mathematical aspects in comparison with practical applications or physical content but there can be no doubt that, by insisting on thorough analysis, by demonstrating the power of mathematical methods and by initiating many younger colleagues into his way of thinking, he has greatly contributed to raising the whole field to a higher level, thus paving the way for future developments.

Eindhoven, September 1960

H. B. G. CASIMIR



Stenographical notes from a lecture by H. A. Lorentz.

Weiss formula leads to: $\sigma_3(n) - \sigma_1(n) + 4\sigma_3\left(\frac{n}{2}\right) + 2\sigma_1\left(\frac{n}{2}\right) =$

$$= 12 \sum_{k=1}^{n-1} \left\{\sigma_1(k) - 2\sigma_1\left(\tfrac{k}{2}\right)\right\} \cdot \left\{\sigma_1(n-k) - 2\sigma_1\left(\tfrac{n-k}{2}\right)\right\}$$

$$\frac{\theta_2^4}{\theta_3^4} = k^2 = 8\int_t^\infty \frac{\theta_0^{12}}{\theta_3^8}\,dt = 8\int_t^\infty \frac{\theta_3^4}{\theta_2^{24}}\,dt = 8\int_t^\infty \frac{\theta_0^4}{\theta_2^{16}}\,dt$$

$$\frac{\theta_3^4}{\theta_2^4} \to 1 \text{ as } t \to 0$$

$$\frac{\theta_3^4}{\theta_2^4} = K_3 \int_0^t \frac{\theta_2^4}{2^{12}\theta_1^{24}}\,dt + 1 = K_3 \int_0^t \frac{\theta_0^{12}}{\theta_2^8}\,dt + 1$$

Hence

$$\frac{\theta_0^4}{\theta_2^4} = K_3 \int_0^t \frac{\theta_2^4}{2^{12}\theta_1^{24}}\,dt$$

In same way for

$$\theta_3^4 = \frac{d}{dt}\,\theta\left(\frac{\theta_0}{\theta_2}\right)^8$$

$$K_3 = 8$$

Hence

$$\frac{\theta_0^4}{\theta_2^4} = 8\int_0^t \frac{\theta_0^{12}}{\theta_2^8}\,dt$$

$$= 8\int_0^t \frac{\theta_2^4\,dt}{2^{12}\theta_1^{24}}$$

Incidentally

$$\varepsilon_2(t) + \varepsilon_3(t) = 2\varepsilon_3(2t) = -\varepsilon_1(t)$$

or:

$$\theta_2^4(0,2t) + \theta_3^4(0,2t) = \tfrac{1}{2}\left(\theta_0^4(t) + \theta_3^4(t)\right)$$

$$\theta_0^4(0,2t) = \theta_0^2(t)\cdot\theta_3^2(t)$$

$$\lambda_2(t) + \lambda_3(t) = \lambda_2(t/2)$$

Dito

$$\frac{\theta_2^4}{\theta_0^4} = 8\int_t^\infty \frac{\theta_0^{12}}{\theta_0^8}\,dt$$

$$= 8\int_t^\infty \frac{\theta_0^4}{\theta_3^{24}}\,dt$$

Notes on elliptic functions.

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Physics. — "*The currents arising in n-coupled circuits when the primary current is suddenly broken or completed.*" By BALTH. VAN DER POL Jr. (Communicated by Prof. W. H. JULIUS.)

(Communicated in the meeting of January 29, 1916.)

Suppose we have two circuits with given resistance and selfinduction and coupled magnetically. If the electromotive force in one of the circuits suddenly stops, the current in it will asymptotically fall to a zero value, whereas the current in the other circuit rises from zero to a maximum value, then gradually falling again to zero.

This paper will treat on the following extension of this problem:

1. The change in time of the currents excited in n equal circuits coupled magnetically, in such a way that the first is coupled with the second, the second with the third, etc., the $n-1^{\text{th}}$ with the n^{th}.

2. The same problem, only the n^{th} being coupled with the first, so as to produce a closed chain of currents.

For the sake of simplicity we will put equal the coefficients of selfinduction and the resistances for all circuits, the same assumption being made for the coefficients of mutual induction.

I. *Linear series of coupled circuits.*

Our case of an electromotive force existing in the first circuit and disappearing suddenly at the time $t = 0$, is analytically equivalent to the case that at the time $t = 0$ the current is zero for all circuits except for the first where its value amounts to $i_1 = \frac{E}{r}$.

Putting the current in the first, second, third, etc. circuit $i_1, i_2, i_3 \ldots$, etc.; and the coefficients of selfinduction L, the coefficients of mutual induction M, the resistances r, then we have the following set of simultaneous differential equations:

$$\left.\begin{array}{ll}
i_1 r + L\frac{di_1}{dt} + M\frac{di_2}{dt} & = 0 \\
i_2 r + L\frac{di_2}{dt} + M\frac{di_3}{dt} + M\frac{di_1}{dt} & = 0 \\
i_3 r + L\frac{di_3}{dt} + M\frac{di_4}{dt} + M\frac{di_2}{dt} & = 0 \\
\cdots\cdots\cdots\cdots\cdots & \\
i_{n-1} r + L\frac{di_{n-1}}{dt} + M\frac{di_n}{dt} + M\frac{di_{n-1}}{dt} & = 0 \\
i_n r + L\frac{di_n}{dt} \qquad + M\frac{di_{n-1}}{dt} & = 0
\end{array}\right\} \quad \ldots \quad (1)$$

the initial condition being

$$\text{for } t = 0 \text{ is: } i_1 = \frac{E}{r},\ i_2 = i_3 = \ldots = i_n = 0.$$

In order to obtain the solution we take

$$\begin{array}{l}
i_1 = \alpha_1 e^{pt} \\
i_2 = \alpha_2 e^{pt} \\
\cdots\cdots\cdots \\
i_n = \alpha_n e^{pt}
\end{array}$$

Further, introducing

$$\frac{r + pL}{Mp} = q \quad \ldots\ldots\ldots \quad (2)$$

and substituting these expressions in the differential equations, we obtain the homogeneous equations