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Nonlinear Evolution Equations and Dynamical Systems

NONLINEAR EVOLUTION EQUATIONS AND DYNAMICAL SYSTEMS

PROCEEDINGS NEEDS '91

Baia Verde, Gallipoli, Italy 19 – 29 June 1991

Editors

M. Boiti, L. Martina and F. Pempinelli

Dipartimento di Fisica, Università di Lecce
Italia

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Preface

The Seventh Workshop on Nonlinear Evolution Equations and Dynamical Systems (NEEDS '91) took place at Baia Verde near Gallipoli, in Southern Italy from June 19 to June 29, 1991. This workshop followed the same pattern, both organizationally and scientifically, as the previous ones, held in Crete (Greece, 1980,1983 and 1989), in Baia Verde (Italy, 1985), in Balaruc les Bains (France, 1987) and Dubna (USSR, 1990). Its main purpose was to bring together, from all over the world, scientists engaged in researches on nonlinear systems, either interested in their underlying mathematical properties or in their physical applications.

A special effort was made to ensure a large attendance by researchers coming from countries with nonconvertible currency. There were 77 participants from 22 countries: Italy (20), USSR(14), France (6), Japan (6), Canada (4), Poland (4), Germany (3), Turkey (3), Belgium (2), Spain (2), USA (2), Australia, Bulgaria, Finland, Holland, India, Korea, South Africa, Sweden, Switzerland, Taiwan, UK (1).

Remarkably, almost all participants gave a lecture: 26 long lectures (45 minutes), 44 short talks (30 minutes) and 6 posters. The topics discussed included integrable, near integrable and nonintegrable evolution equations and dynamical systems. The talks ranged from pure mathematics through numerical computations and applications to various field of physics. In addition to the scheduled program, many informal exchanges of ideas and free discussions enriched the workshop.

This volume includes in written form most of the talks given at the meeting. So it is devoted to current research in nonlinear evolution equations and dynamical systems. In our opinion, for the large variety of topics that have been covered, and for the quality of the contributions, these proceedings give a good up-to-date picture of the state of art in the field. They do not provide an exhaustive self-contained description of the whole subject, but rather give an outline of the most recent and relevant results in such a way that they should stimulate the interested reader.

The NEEDS series of workshops demonstrate once more that broad international collaboration is effective and fruitful in the Nonlinear Science field. This subject benefits from the cooperation of specialists, working in fields ranging from pure mathematics to the applied science. These features were underlined during the presentation of the European Institute for Nonlinear Studies via Transnationally Extended Interchanges (EINSTEIN), created in Lecce (Italy), with the aim to pursue 1) the organization of international meetings and long duration workshops in the realm of Nonlinear Science, 2) the exchanges of scientists coming mainly from Western countries, Eastern Europe and the countries of the ex-USSR, 3) to stimulate fast exchanges of information, as well as to perform jointly large computational tasks via efficient computer connections.

The Workshop NEEDS '91 was organized by researchers from the University of Lecce (Italy), and they would like to thank the creator of the NEEDS series, Prof. F. Calogero from the University of Rome, for his continuous encouragement. The meeting was sponsored by the University of Lecce and by the Istituto Nazionale di Fisica Nucleare (INFN), Italy.

The organizers took advantage of the services of the Dipartimento di Fisica of Lecce University. They wish in particular to thank Mr F. Spagna, Mrs A. Vergori, who all together took care of the good working of the meeting, and in particular way Mrs M. C. Gerardi, who actively participated in the organization of the workshop. The Editors of the present Proceedings wish to thank all the authors, who sent their contributions. The original style of presentation has been preserved, and only minor misprints have been corrected where possible. Finally the Editors wish to aknowledge Mr Gino Pastore for the beautiful drawing of the cover.

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M. Boiti
L. Martina
F. Pempinelli

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Part I

Integrable Systems in Multidimensions

SELF DUAL YANG-MILLS EQUATION AND NEW SPECIAL FUNCTIONS IN INTEGRABLE SYSTEMS

S. CHAKRAVARTY, M. J. ABLOWITZ
Program in Applied Mathematics, University of Colorado
Boulder, CO 80309 USA.

and

L. A. TAKHTAJAN
Program in Applied Mathematics, University of Colorado
Boulder, CO 80309 USA.
and
Steklov Mathematical Institute, nab. Fontanka 27
St. Petersburg, 191011 USSR.

ABSTRACT

The Self Dual Yang-Mills equations admit reductions to many of the well known integrable soliton systems. By allowing the gauge potentials to be elements of suitable infinite dimensional Lie algebras a novel class of nonlinear systems are obtained as reductions of the self dual equations. Some of these new systems themselves have reductions to the classical Chazy equation which has as its only movable singularities, a natural boundary.

1. Introduction

In recent years much work has been devoted to the intriguing possibility that the Self-Dual Yang-Mills (SDYM) equations and its generalizations may be viewed as a "master equation" in which the well known soliton equations are embedded. Indeed Ward[1] made such a conjecture and showed that some of the well known integrable nonlinear equations arising in widely different areas of mathematical physics could be obtained by a reduction of variables of the SDYM equations with appropriate choices of the gauge group in which the gauge potentials reside. Subsequent work by several authors[2,3] have shown that the equations obtained as reductions of SDYM equations is quite large, and that virtually all of the "classical" soliton equations can be obtained.

Significantly, new possibilities arise when the potentials are allowed to lie in an infinite dimensional Lie algebra. In this note we discuss a novel system, regarded as an extension of the well known Nahm[4] equations, that is obtained from the SDYM equations associated with a particular infinite dimensional Lie algebra, namely the volume preserving diffeomorphisms on S^3. In 0+1 reduction, this new system is related to a classical, third order, nonlinear differential equation studied by Chazy[5]. This equation has movable natural boundaries in the complex-plane and consequently adds to the richness of the exactly solvable systems. As such it is different from the soliton systems since, generally speaking, reductions of exactly solvable equations are of Painleve type, i.e., their movable singularities are poles. Moreover, it can be shown that the solutions of the Chazy equation may be written in terms of modular forms. The new systems we have found can be expected to have solutions which are in a sense generalized modular forms. This, in a way, is analogous to how Riemann theta functions of finite genus generalize the classical Jacobi theta functions as solutions of the underlying soliton systems.

2. Preliminaries

It is standard to formulate the YM equations in terms of the curvature,

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a , A_b] \tag{1a}$$

where $\partial_a = \partial/\partial x^a$, $x^a = \{t = x^0, x^i\}$, $i = 1,2,3$ being the coordinates in Euclidian space $\mathbf{E}^4$. The A_a's are the YM connection 1 - forms (gauge potentials) and take values in some Lie algebra. The SDYM field equations are given by

$$F_{oi} = \frac{1}{2}\sum_{j,k} \varepsilon_{ijk} F_{jk} \tag{1b}$$

where ε_{ijk} is an alternating tensor with $\varepsilon_{123} = 1$. Eq. 1(b) follows from the standard definition of duality in $\mathbf{E}^4$. Note that the SDYM equations (1b) are invariant under the gauge transformation : $A_a \to gA_a g^{-1} - (\partial_a g)g^{-1}$ for any differentiable function $g = g(x^a)$ taking values in the corresponding Lie group. Eq.1(b) can be obtained as the integrability condition of a pair of linear PDE's parametrized by a complex spectral parameter – commonly referred to as the Lax pair.

3. One Time (0+1) Reductions

We let the A_a's to be *only* functions of the t (time) coordinate.Then the available gauge freedom allows us to choose the time-component of the vector potential $A_o = 0$ by a suitable choice of the function $g = g(t)$ mentioned in Section II.The reduced SDYM equations take a very simple form when expressed in terms of the A_a's as can be easily seen by using Eqs.1(a) and 1(b)

$$\partial_t A_i = \frac{1}{2}\sum_{j,k} \varepsilon_{ijk}[A_j, A_k] \tag{2}$$

These ODE's are known as the Nahm equations when the A_a's belong to a simple Lie algebra e.g. su(n). They were introduced by Nahm[4] to construct static solutions of non-abelian magnetic monopoles in $\mathbf{E}^3$. Specifically, we will take the A_i's in the Lie algebra su(2) generated by the traceless skew-hermitian matrices σ_i, $i = 1,2,3$ satisfying the commutation relations: $[\sigma_i , \sigma_j] = \sum \varepsilon_{ijk}\sigma_k$. We will give *two* simple examples of su(2) - reductions of Eq.2 which will be helpful in our later discussions regarding the extensions of these equations for infinite dimensional Lie algebra.

Example I :

The first system of equations are obtained by setting $A_i = \omega_i \sigma_i(t)$ (no sum) in Eq.2 and using the commutation relations for the σ_i's:

$$\partial_t \omega_1 = \omega_2 \omega_3, \quad (3a)$$

$$\partial_t \omega_2 = \omega_1 \omega_3, \quad (3b)$$

$$\partial_t \omega_3 = \omega_2 \omega_1 \quad (3c)$$

This system of equations can be transformed easily to the Euler equations of motion for a rigid body with no external force. In fact, it is straight forward to see that Eq.3 can be integrated. From Eqs. 3(a) and (b) we see that the quantity $K = \omega_1{}^2 - \omega_2{}^2$ is a constant. Next, introduce a function $\phi(t)$ such that

$$\omega_1 = K \cosh\phi, \qquad \omega_2 = K \sinh\phi \quad (4)$$

and substitute Eq.4 in either of Eqs. 3(a) or (b). We have that ω_3 can be expressed as $\omega_3 = \partial_t \phi$. Using this in Eq.3(c) one obtains a second order ODE for ϕ

$$\partial_{tt}\phi = (K^2/2)\sinh(2\phi) \quad (5)$$

which then can be integrated to obtain solutions for the ω_i's in terms of Jacobi elliptic functions. Thus the solutions for Eqs.3 are essentially the same as that of the Euler "top" equations.

Example II :

There exists another interesting reduction of the su(2) Nahm equations if one makes the following choice for the vector potentials

$$A_1 = a\sigma_1 + b\sigma_2, \quad A_2 = -b\sigma_1 + a\sigma_2, \quad A_3 = c\sigma_3 \quad (6)$$

where a, b, and c are functions of t to be determined and the σ_i's satisfy the above commutation relations. Substitution of Eq.6 in Eq.2 yields the following set of ODE's for the functions a, b, and c

$$\partial_t a = ac \quad 7(a)$$

$$\partial_t b = bc \quad 7(b)$$

$$\partial_t c = a^2 + b^2 \quad 7(c)$$

A similar calculation as that of Eq. 5 shows that Eqs. (7) can be reduced to a single second order ODE as in Example I

$$\partial_{tt} u = K e^{2u} \quad (8)$$

with K constant. The functions a,b, and c can be expressed in terms of u(t) as

$$a(t) = Ae^u, \qquad b(t) = Be^u, \qquad c(t) = \partial_t u \quad (9)$$

where A, B are constants and $K = A^2 + B^2$. Eq.8 can be integrated in quadratures and solutions for a,b and c can be represented in terms of trigonometric functions.

We summarize our discussions above by the following remarks:

1. The corresponding system of equations (Eqs.3 and 7) admit complex analytic solutions which are either singly periodic (trigonometric functions) or doubly periodic (elliptic functions) and meromorphic i.e., the only singularities are isolated simple poles in the complex t - plane.

2. The examples of Eq.2 considered above only involve su(2) as the underlying gauge algebra. The situation is not dramatically different if one considers other finite dimensional

Lie algebras. For example, it has been shown by Hitchin[6] that if the A_i's in Eq.2 are in the Lie algebra su(n) and satisfy certain conditions, then their solutions are meromorphic in the complex t-plane and can be represented as ratios of Jacobi theta functions.

3. It is also possible to construct the solutions of the Nahm equations associated with finite dimensional Lie algebras by using the global analytic solution of the underlying Lax pair. These solutions are defined on a Riemann surface of finite genus often referred to as the "spectral curve". The spectral curve corresponding to su(2) Nahm equations is an elliptic curve.

The situation is completely different when the A_i's are allowed to lie in an *infinite dimensional* Lie algebra. In particular, we consider the Lie algebra of volume preserving diffeomorphism of a 3-sphere – sdiff(S^3). We devote the remainder of this section to another SDYM reduction which we refer to as the Halphen system[7]. We will show that the resulting equations are still integrable but the solution structure is entirely different from the systems discussed above. The Halphen equations can be reduced to a third order, nonlinear differential equation which admits a natural boundary in the complex t-plane and their solutions can be represented in terms of automorphic forms of fixed weights defined by their transformations under the group SL(2,C) acting projectively onto the complex t-plane.

It is convenient to coordinatize S^3 in terms of the Euler angles: $0 \le \theta \le \pi$, $0 \le \phi \le \pi$, and $0 \le \psi \le 4\pi$. The particular generators of sdiff(S^3) that we choose are the vector fields:

$$\begin{aligned} X_1 &= \cos\psi\partial_\theta + (\sin\psi/\sin\theta)\partial_\phi - \cot\theta\sin\psi\partial_\psi\,, \\ X_2 &= -\sin\psi\partial_\theta + (\cos\psi/\sin\theta)\partial_\phi - \cot\theta\cos\psi\partial_\psi\,, \\ X_3 &= \partial_\psi \end{aligned} \tag{10a}$$

The X_i's are called the rotational Killing vectors of $S^3 = SU(2)$, i.e., they leave the standard metric on S^3 and the compatible volume form (the Haar measure on SU(2)) invariant under their action. Furthermore, one can directly verify from their defining relations – Eq.10(a) that the X_i's satisfy the same commutation relations as the generators of the Lie algebra so(3)

$$[X_i, X_j] = \sum_k \varepsilon_{ijk} X_k \tag{10b}$$

where the bracket denotes the Lie derivative for vector fields. It is very important to note however, that the two Lie algebras are entirely different. The generators of so(3) form a three dimensional vector space whereas the linear combinations of the X_i's with coefficients which are arbitrary *functions of the Euler angles*, span an *infinite* dimensional Lie algebra.

The A_i 's are expressed as

$$A_i = \sum_j P_{ij}(t,\theta,\phi,\psi) X_j \tag{11a}$$

The matrix P is chosen to be the product

$$P = O(\theta,\phi,\psi) M(t) \tag{11b}$$

where, $O \in SO(3)$ is the usual rotation matrix and M(t) is a 3×3 matrix of field variables. Substituting the A_i's from Eq.11(a) in Eq.(2) we get

$$\sum_l P_{il}\frac{dA_l}{dt} = \frac{1}{2}\sum_{j,k,r,s} \varepsilon_{ijk} P_{jr} P_{ks} [A_r, A_s] + \sum_{j,k,r,s} \varepsilon_{ijk} P_{jr} A_r(P_{ks}) A_s \tag{12a}$$

The second term of the rhs of Eq.12 is the contribution from the Lie derivative due the action of the vector fields X_i 's on $O_{ij}(\theta,\phi,\psi)$ which is given by