OPERATOR THEORY

AND

GROUP REPRESENTATIONS

Report of an International Conference

on

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Foreword

On October 20–23, 1953 the National Academy of Sciences—National Research Council and the International Mathematical Union sponsored a conference on *Operator Theory and Group Representations*, held at Arden House, Harriman, New York.

A record of the addresses delivered at the various sessions follows:

Tuesday, October 20 (evening):

HALMOS, P. R., Subnormal operators
WERMER, J. Restrictions of operators

Wednesday, October 21 (morning):

MACKEY, G. W., The present status of the theory of group representations
SEGAL, I. E., Non-commutative integration

Wednesday, October 21 (afternoon):

MAUTNER, F. I., Ergodic double cosets
ARENS, R. F., Group algebras of ordered groups
HELSON, H., Bounded groups in algebras of measures

Thursday, October 22 (morning):

KADISON, R. V., The full linear group of a factor
SINGER, I. M., The automorphism group of a factor
DYE, H. A., Unitary groups in a factor

Thursday, October 22 (afternoon):

HARISH-CHANDRA, Representations of Lie groups

Thursday, October 22 (evening):

KAPLANSKY, I., Modules over operator algebras
LOOMIS, L., Lattices and rings of operators

Friday, October 23 (morning):

HEINZ, E., Inequalities for operators
RELLICH, F., Linearly perturbed operators

The organizing committee is very grateful to Professor Kadison for his report on Operator Algebras; to Professor Singer for his report on Group Representations; to Professor Wermer for his report on Subnormal Operators; and to Dr. Heinz and Professor Rellich for their contributed papers.

G. W. MACKEY
M. H. STONE
I. KAPLANSKY, *Chairman*

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Report on Subnormal Operators

By J. Wermer

Let A be a bounded normal operator on a Hilbert space $\mathfrak{J}$ and let $\mathfrak{H}$ be a closed subspace of $\mathfrak{J}$ invariant under A. The restriction B of A to $\mathfrak{H}$ is then a bounded operator on the Hilbert space $\mathfrak{H}$. Halmos calls B *subnormal.*

The following questions now arise:

1. When is a given operator B subnormal, i.e., when can its space of definition $\mathfrak{H}$ be embedded in a larger space $\mathfrak{J}$ so that B turns out to be the restriction of some normal operator on $\mathfrak{J}$?
2. What is a concrete representation of B?
3. What is the relation of the spectrum of B to that of the normal operator A?
4. Does B always have non-trivial invariant subspaces?

Question 1 was answered by Halmos in [1] with the following theorem (simplified in its hypotheses by Joseph Bram):

B is subnormal if, and only if, for every set $x_0, x_1, \cdots x_n$ of vectors in $\mathfrak{H}$, the matrix whose general entry is $(B^k x_j, B^j x_k)$ is positive.

Suppose next that $\mathfrak{H}$ has a single generator under B, i.e., for some x in $\mathfrak{H}$, the vectors $B^n x$, $n \geqq 0$ span $\mathfrak{H}$. Let E be the spectral measure for the normal extension of B and let $d\mu = d \| E_\lambda x \|^2$. Then μ is a positive measure with compact carrier. Let H_μ be the closed subspace of the L^2-space formed from μ which is spanned by polynomials. Then B is equivalent to the operator on H_μ of multiplication by λ. (This was pointed out by Singer.)

The structure of the space H_μ is however in general not at all evident and this representation does not yield an answer to question 4.

In answer to question 3 Halmos showed in [2]: *If A is normal on $\mathfrak{J}$, B the restriction of A to $\mathfrak{H}$, and if there is no proper subspace of $\mathfrak{J}$ which contains $\mathfrak{H}$ and reduces A, then the spectrum of B (as operator on $\mathfrak{H}$) includes the spectrum of A (as operator on $\mathfrak{J}$).*

Question 4 was given a partial answer by Wermer as follows. Let A be a normal operator, $\sigma(A)$ its spectrum, $\rho(A)$ its resolvent set, R_λ its resolvent and let E be the spectral measure of A.

THEOREM. *Let $\sigma(A)$ have two-dimensional Lebesgue measure null. Then for every restriction B of A to a subspace $\mathfrak{H}$, B has a non-trivial invariant subspace in $\mathfrak{H}$.*

PROOF. Set $\mathfrak{H}^\infty = \{y \text{ in } \mathfrak{H} \mid R_\lambda y \in \mathfrak{H} \text{ for all } \lambda \text{ in } \rho(A)\}$. Then clearly $\mathfrak{H}^\infty$ is a closed subspace of $\mathfrak{H}$, invariant under A.

We claim A^* also leaves $\mathfrak{H}^\infty$ invariant. For else there is some y in $\mathfrak{H}^\infty$, $A^*y \notin \mathfrak{H}^\infty$. Hence there exists g, $g \perp \mathfrak{H}^\infty$, $(A^*y, g) \neq 0$. But since $y \in \mathfrak{H}^\infty$, $R_\lambda y \in \mathfrak{H}^\infty$ for each λ and so

$$(R_\lambda y, g) = \int_{\sigma(A)} \frac{1}{\xi - \lambda} d(E_\xi y, g) = 0 \quad \text{for} \quad \lambda \quad \text{in} \quad \rho(A).$$

Hence

$$\int r(\xi)\, d(E_\xi y, g) = 0$$

for every rational function with all its poles in $\rho(A)$. By [3], every continuous function on $\sigma(A)$ is uniformly approximable by such rational functions on $\sigma(A)$, since $\sigma(A)$ has measure zero.

Hence $d(E_\xi y, g)$ is the null measure, whence

$$(A^*y, g) = \int \bar{\xi}\, d(E_\xi y, g) = 0.$$

This is a contradiction. Hence $A^*\mathfrak{H}^\infty \subseteq \mathfrak{H}^\infty$, as asserted.

Thus B is normal on $\mathfrak{H}^\infty$. If then $\mathfrak{H}^\infty = \mathfrak{H}$, B is normal on $\mathfrak{H}$ and so has non-trivial invariant subspaces. If $\mathfrak{H}^\infty \neq \mathfrak{H}$, $\exists z$ in $\mathfrak{H}$, $z \notin \mathfrak{H}^\infty$. Hence for some λ_0 in $\rho(A)$, $R_{\lambda_0} z \notin \mathfrak{H}$.

Set $C = \{y \in \mathfrak{H} \mid R_{\lambda_0} y \in \mathfrak{H}\}$. C is a closed subspace of $\mathfrak{H}$, $\neq 0$ and invariant under B. Also $z \notin C$, and so $C \neq \mathfrak{H}$. Thus B has a non-trivial subspace in this case also.

Subnormal operators share some properties with more general operators obtained by restriction. (Cf. Wermer, [4].)

Let τ be a bounded operator with inverse τ^{-1} defined on an arbitrary Banach space and C a closed τ-invariant subspace, and let T denote the restriction of τ to C.

THEOREM. *Suppose C has a single generator under T. Then C is homomorphic to a space of analytic functions defined on part of the λ-plane, T being represented as multiplication by λ. The kernel of this homomorphism is*

$$C^\infty = \{y \in C \mid \tau^{-n} y \in C,\ n = 1, 2, \cdots\}$$

COROL. *If $C^\infty = (0)$, then the ring of all operators commuting with T is isomorphic to a ring of bounded analytic functions.*

Take now τ in the preceding theorem to be a normal operator on Hilbert space, and suppose $\sigma(\tau)$ has measure 0, and $\rho(\tau)$ has two components, D denoting the bounded component. Then, if T again means the restriction of τ to C we have:

THEOREM. *C decomposes into C^∞ and its orthogonal complement C^0. Both C^∞ and C^0 are invariant under T. On C^∞, T is normal, while C^0 is isomorphic to a Hilbert space $\mathfrak{F}$ of analytic functions on $\mathfrak{D}$ and T is represented on $\mathfrak{F}$ as multiplication by λ.*

This theorem raises the question:

When is $C = C^\infty$ and when is $C = C^0$?

For the case when τ is unitary, Kolmogoroff in [5] gives a necessary and sufficient condition on the generating vector x of C and the spectral measure of τ in order that $C = C^0$. In this case, furthermore, all closed invariant subspaces of the subnormal operator T on C^0 were determined by Beurling in [6]. For certain other normal operators τ conditions were given by Wermer in [7] assuring that $C = C^\infty$ for every restriction of τ.

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Report on Operator Algebras

By Richard V. Kadison

The Arden House Conference was concerned with recent research in operator theory, group representations, and their interconnections. In addition to the formal twenty and forty minute addresses, there was considerable informal discussion. It would be difficult to report accurately on the informal portion of the conference and the results arising therefrom.

This section of the conference report will be devoted to a survey of the theory of operator algebras and to the relation of those formal addresses which dealt primarily with operator algebras to the broader aspects of this theory. We shall omit all bibliographical references, since we could not hope to include reference to all the papers which have directly contributed to the state of our present knowledge about operator algebras in a report of reasonable size.

A C*-algebra is an algebra of operators on a Hilbert space which is closed under the operation of taking the adjoint and closed in the operator bound (uniform) topology. A C*-algebra is the natural infinite-dimensional analogue of a finite-dimensional algebra of complex matrices closed under the operation of taking the conjugate transpose (the topological conditions which might be imposed, in the finite-dimensional case, are automatically satisfied by virtue of the finite-dimensionality). These finite-dimensional matrix algebras are, of course, special cases of C*-algebras. Their structure is completely described by the Wedderburn theory (algebraically they are direct sums of total complex matrix algebras of various orders and, with regard to their specific action on the underlying space, they are direct sums of n_i-fold copies of total matrix algebras of order m_i, $i = 1, \cdots, k$). In general terms, the central problem in the study of C*-algebras is that of finding a structure theory which will do for these algebras what the Wedderburn theory does for the finite-dimensional C*-algebras.

Aside from their intrinsic interest as a natural class of infinite-dimensional, semi-simple algebras, C*-algebras find application in the study of group representations, mathematical formulations of physical situations, and certain phases of ergodic theory. If we denote by G a locally compact group (assumed unimodular, for the sake of simplicity), by $L_1(G)$, $L_2(G)$ the integrable and square integrable functions on G relative to Haar measure, respectively, then the functions of $L_1(G)$ acting by (left) convolution on $L_2(G)$ give rise to a family of bounded operators on the (Hilbert) space $L_2(G)$ closed under the adjoint operation. This family of operators and its closures (all of which are C*-algebras) in the various operator topologies serve as generalizations of the complex group algebra of a finite group. These group algebras play a crucial role in the study of the group representations of G. The measure-theoretic properties of groups of measurability preserving transformations on a measure space can be studied by investigating the structure of the various C*-algebras obtained from the operators derived from the action of the group on the Hilbert space of square integrable functions over the measure space together with the operators arising from multiplication by essentially bounded measurable functions on this space of square integrable functions.

The methods used in the study of C*-algebras are quite diverse. Of course, the techniques derived from modern algebra are employed extensively. While algebraic techniques are sufficient, almost exclusively, for dealing with the finite-dimensional situation, they don't begin to give the full picture in the case of infinite-dimensional C*-algebras. The continuous as well as the discrete (e.g., with regard to spectra) arises in the infinite-dimensional case, while it is not present in the finite-dimensional situation. These considerations make the tools of analysis, notably, complex function theory and abstract measure theory, invaluable for the investigation of C*-algebras. In addition to these methods, a special brand of point set topology which fashions a topological structure to the algebraic and intrinsic geometrical structure has proved quite useful in the study of C*-algebras.

It should be remarked that we seem to be not too close to a final structure theory for C*-algebras. We have no guesses as to how the general C*-algebra is constructed from a "canonical set" of fully understood C*-algebras. Aside from this lack of a general theory, however, the subject bristles with simply phrased, quite specific, "yes" or "no" questions for which we have neither the answer nor reasonable guesses as to the answer.

A well-known result of Gelfand-Neumark tells us that a C*-algebra has an independent algebraic existence, viz., a Banach algebra with a *-operation having the usual formal algebraic properties and satisfying, in addition, $\| aa^* \| = \| a \|^2$ is isometrically *-isomorphic with a C*-algebra. Some years ago, M. H. Stone proved a theorem about commutative C*-algebras which gave the algebraic portion of the spectral theorem a very cogent form. He showed that each commutative C*-algebra is algebraically isomorphic to the algebra of all continuous, complex-valued functions on some compact-Hausdorff space (derived from the algebraic structure of the C*-algebra) with the *-operation in the C* algebra going into complex-conjugation of functions. He showed, moreover, that the C*-algebra is determined to within algebraic isomorphism by the homeomorphism type of the associated compact-Hausdorff space. The function ring on each compact-Hausdorff space is easily seen to be a (commutative) C*-algebra, so that the distinct classes of algebraically isomorphic, commutative C*-algebras are in 1-1 correspondence with the homeomorphism classes of compact-Hausdorff spaces. For the purposes of operator theory, this is an adequate algebraic description of such operator algebras. To a non-commutative C*-algebra, one can again associate a structurally derived compact-Hausdorff space and, this time, a distinguished linear subspace of continuous, complex-valued functions on this compact-Hausdorff space. However, in this case, we do not know canonical forms for the linear subspace taken together with the compact-Hausdorff space, although the system characterizes the C*-algebra.

A commutative C*-algebra together with its action on its underlying Hilbert space can be described by its associated compact space and a well-ordered chain of ideals of Borel sets in the space (each, the family of null sets of some measure). Again, we do not have canonical forms for such a construct, but the problems involved in obtaining such canonical forms are in the province of pure measure theory and are already inherent in the classical unitary equivalence description of the action of a single self-adjoint operator on a Hilbert space by Hellinger-Hahn (of which the commutative C*-algebra result is an extension). Aside from the original Hellinger-Hahn theory, Wecken, Plessner, Rohlin, Segal, Nakano, and Halmos have contributed important techniques to this final formulation of commutative multiplicity theory. It has become possible, recently, to make an analogous study of the action of a not necessarily commutative C*-algebra on its underlying Hilbert space, assuming the algebraic structure known. This theory inherits, of course, all the problems of the commutative theory, but seems, at this stage, to have no others.

The class of C*-algebras has several important subclasses which have received special attention. Notable among these is the class of "rings of operators" (also called "W*-algebras"—those closed in the weak operator topology, i.e., the weakest (coarsest) topology on the bounded operators in which all the linear functionals of the form $A \to (Ax, y)$ are continuous). The assumption that a C*-algebra be weakly closed produces deep effects upon its structure, and the additional algebraic and geometrical properties visible enable us to subject this class of C*-algebras to a much more detailed analysis (though, by no means, a definitive analysis, at this point of development of the subject). In particular, rings of operators (containing the identity operator) contain, along with each self-adjoint operator, its complete spectral resolution. J. von Neumann has exhibited rings of operators as "direct integrals" (measure-theoretic generalization of direct sum) of basic constituents called "factors" (rings of operators whose center consists of scalar multiples of the identity operator). Murray and von Neumann have studied these factors in detail. By comparing the relative sizes of the ranges of orthogonal projections in a given factor, M, a relative dimension function D is defined on the projections in M (having the customary properties of a dimension function) and is shown to be unique to within a positive multiplicative constant. With the aid of this dimension function, the factors are separated into three classes. The first class comprises the factors of type I_n, those having minimal projections in which the (normalized) dimension function takes the values $1, 2, \cdots, n$ (n finite or infinite). The second class constitute the factors of type II_1 and II_∞ in which the dimension function takes all values in $[0, 1]$ and $[0, \infty]$, respectively. These are the factors having no minimal projections and containing a non-zero projection of relative dimension different from ∞. The final class consists of the factors of type III in which the dimension of each non-zero projection is ∞. The factors of type I_n are shown to be algebraically *-isomorphic to the algebra of all bounded operators on an n-dimensional Hilbert space. Associated with each factor on a Hilbert space, one has the set of operators which commute with it, which is again a factor of type I, II, or III, according as the original factor is of type I, II, or III, respectively. If M is of type I_n, M' (the commutant of M) of type I_m, N is of type

I_n and N' of type I_m, then M and N are unitarily equivalent. In general, if M is a factor of type I or II with commutant M' there is associated with M a constant, the so-called "coupling constant". If x is a non-zero vector in the underlying Hilbert space upon which M acts, the orthogonal projections E and E' on the closures of the linear manifolds spanned by the images of x under operators in M' and M, respectively, lie in M and M', respectively. The ratio of the dimensions of E and E', relative to M and M', respectively, is the coupling constant just mentioned (it is shown to be independent of the vector x chosen). If N is another factor algebraically *-isomorphic to M, with commutant N' and coupling constant equal to that of M and M', then M and N are unitarily equivalent, and, moreover, the given algebraic isomorphism can be implemented by a unitary transformation. This result does not apply *per se* to the case where M is of type II_∞ and M' of type II_1. This last case can be handled, however, by suitable modifications of the above mentioned techniques. Recently, E. L. Griffin has shown that (at least in the case of separable Hilbert space) each *-isomorphism between factors of type III can be implemented by a unitary transformation between the underlying Hilbert spaces. The problems then, in the study of factors and rings of operators, are largely ones of the algebraic nature of these operator algebras.

By considering the weakly closed group algebras of various locally compact topological groups, examples can be constructed of each of the various types of factors. In point of fact, however, factors of type III were constructed, only after much effort, by considering groups of measurability preserving transformations acting on measure spaces which do not admit group invariant measures.

In terms of the dimension function constructed, a trace function with the usual properties can be introduced in factors of types I_n and II_1. In terms of this trace function a topology can be imposed on the factor which is useful for the study of its structural properties.

Current research in the theory of operator algebras centers about the study of factors of type II_1. A broad class of factors of type II_1, the socalled "approximately finite factors" in which any finite set of operators can be approximated as closely as desired in the trace topology by operators lying in a subring of finite linear dimension, have been shown to be algebraically *-isomorphic to each other. On the other hand, it has also been shown that there are factors of type II_1 which are not of the same algebraic type as the approximately finite factors. This is effected by showing that the approximately finite factors of type II_1 possess an approximate (relative to the trace topology) commutativity property which the weakly closed group algebra of certain groups does not have (e.g., the free group on two generators).

With regard to the study of factors and, more generally, rings of operators, one of the important projects involves the analysis of the structure preserving maps. At the Arden House Conference, I. M. Singer presented some of his recent results concerning the automorphisms of factors of type II_1. He considered factors of type II_1 arising from groups of measure-preserving transformations acting ergodically upon a finite, non-atomic measure space. Roughly speaking, the measure preserving transformations induce unitary operators on the Kronecker product of the Hilbert space of square integrable functions on the group with the Hilbert space of square integrable functions on the measure space. This group of unitary operators taken together with the algebra A of operators obtained from the multiplication action of essentially bounded measurable functions on the measure space generate a factor M of type II_1. The subalgebra A of M can easily be shown to be a maximal abelian subalgebra of M. Singer studies the group G of *-automorphisms of M which leave A set-wise-invariant and its normal subgroup G_0 consisting of those automorphisms in G which leave A elementwise invariant. He describes G in terms of the original group of measure-preserving transformations. In particular, he proves that G is the semi-direct product of G_0 and another group described in terms of the original constructions. A neat statement of these results in cohomological terms was presented. By these means, Singer can show that, in many cases, where the outer automorphisms themselves are not apparent, the factor in question must admit *-automorphisms which are not inner. The present author had raised the question of whether or not a factor of type II_1 (or, more generally, a ring of operators) obeys some sort of Galois theorem relative to its group of *-automorphisms (such is the case for rings of type I). Singer answers this question negatively on the basis of his general techniques with specific examples.

Relating to the question of structure preserving maps of operator algebras, I. Kaplansky presented results concerning derivations of certain classes of C*-algebras. It is appropriate, at this point, to note another trend in current research on operator algebras. Various subclasses of C*-algebras more accessible, structurally, than the general C*-algebra are considered. One of the main proponents of this approach is I. Kaplansky who has developed a reasonably detailed structure theory for a class of C*-algebras he calls "CCR algebras" (those admitting sufficiently many representations by algebras of completely continuous operators). He has introduced a class of algebras he calls AW*-algebras (abstract W*-algebras). This class of C*-algebras embodies the main algebraic features of W*-algebras while being algebraically defined (it is a broader class than the W*-algebras). Kaplansky and others have pursued the program of carrying over to the AW*-algebras the known algebraic properties of W*-algebras, as well as trying to extend the known theory of W*-algebras in terms of AW*-algebras. In his conference talk, I. Kaplansky introduced a construct which he calls a "C*-module". It is a module with an abelian C*-algebra as operator ring and an "inner product" with values in the abelian C*-algebra. This construct may prove to be a very convenient tool for the investigation of operator algebras and for providing new examples of operator algebras (especially, if the general theory can be extended to not necessarily commutative rings of operators). Kaplansky discussed the general theory of his C*-modules but specialized, in a short time, to the case where his C*-algebra was an AW*-algebra and the module over this algebra satisfies two additional algebraic assumptions. Such C*-modules, he calls AW*-modules, and, for these, he carries the general theory much further. With the aid of this new device, Kaplansky then settles an open existence question for certain classes of AW*-algebras. Among other things, he proves that each derivation of an AW*-algebra of type I is inner, basing his argument on a lemma due to Singer.

The study of factors leads one to the study of various algebraic structures attached to these factors. In particular, the group of all unitary operators in a factor and the group of all invertible operators in a factor have attracted a certain amount of attention recently. Henry A. Dye talked on the unitary group in a factor of type II_1, and showed that certain isomorphisms between the unitary groups of such factors give rise to *-isomorphisms or *-anti-isomorphisms between the factors. In this connection, I. Singer has shown that a Lie algebra isomorphism between factors of type II_1 satisfying certain slight continuity conditions implies the existence of a *-isomorphism between the factors. I. Kaplansky has relaxed these conditions somewhat. The present author talked on the structure of the unitary and general linear groups of a factor. A complete list of the uniformly closed normal subgroups was given. It might be remarked that these groups are a natural generalization of the classical groups. L. Loomis talked on a general ordered structure resembling the order structure of the projections in a factor. For such structures, Loomis is able to develop a dimension theory, but, without the added structure of a factor, his techniques must be more delicate than those employed by Murray and von Neumann to define a dimension function on factors.

Another important trend in current research on operator theory is the global investigation of rings of operators. As noted earlier, a ring of operators admits a type of measurable decomposition into factors, relative to its center. This focuses attention on the study of factors. In reality, however, the passage from information about the factors in a decomposition to information about the ring from which they derive is rarely smooth, involving, as it generally does, thorny difficulties of a measure-theoretic and operator-theoretic nature. Since rings rather than factors arise in applications, it is desirable to have some global techniques for dealing with them rather than passing to the factor decomposition. Dixmier, Dye, Godement, Griffin, Kaplansky, Segal, and others have developed such techniques. Dixmier systematically investigated the center-valued trace in rings of operators. Kaplansky's work on AW*-algebras contributed heavily to our global techniques. The methods used are a rather interesting mixture of classical measure theory and modern operator theory, which have their roots in the early work of Murray and von Neumann. I. E. Segal formalized this interrelation between measure theory and operator theory in a non-commutative integration theory. It should be noted that the measure space rather than the range of values is the non-commutative object (the measurable sets corresponding to the projections in a ring of operators and the integration process corresponding to a trace like linear functional). Surmounting considerable technical difficulties, Segal proves non-commutative analogues to the Riesz-Fischer and Fubini Theorems as well as other classical measure-theoretic theorems. At the conference, Segal talked on an extension of dimension theory to arbitrary rings of operators without a finiteness assumption. He discussed a cardinal-valued integration theory appropriate to this extension. Segal also discussed non-commutative extensions of probability theory. He defined a (not necessarily commutative) abstract probability space and proved, among other things, the (non-commutative) analogue of the Kolmolgoroff theorem concerning the existence of random variables having preassigned joint distributions (satisfying certain necessary consistency conditions). In the process, Segal, gives a systematic treatment of direct limits of rings of operators.

It would be rash to say that we are confident of an early solution to the central problems still facing us. Nevertheless, though these problems seem quite difficult and recent progress slow, many of us have hope for a useful structure theory for self-adjoint operator algebras in the not too distant future.

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Report on Group Representations

By I. M. Singer

1. Introduction. The group representation section of the Arden House conference consisted of five lectures, three on group representations proper and two on related topics. Specifically, Mackey gave a partially expository lecture on the present status of group representations, Harish-Chandra described a Plancherel formula for complex semisimple Lie groups, and Mautner discussed geodesic flows on manifolds. Helson spoke on isomorphisms between group algebras and Arens on group algebras of ordered groups.

We feel that the clearest discussion of the talks given can be carried out in the context of a general review of group representations. Much of this review is based on an outline given to us by Mackey of his talk. We claim sole responsibility for the omissions and errors which may have occurred. In particular, we have not discussed explicitly spherical functions, generalized trace functions, and positive definite functions. We wish to point out that the bibliography contains only those papers which bear directly on the contents of this general review and omits many papers which have played a key role in the development of the subject of group representations.

2. The basic problem. A unitary representation of a locally compact group G is a continuous homomorphism of G into the group of unitary operators on a Hilbert space, topologized in the strong operator topology. One deals with infinite dimensional representations because in general a locally compact group need not have any non-trivial finite dimensional bounded representations, and one deals with unitary representations because they are easier to handle and there are sufficiently many irreducible ones [31, 85].

Two unitary representations U_i of G on Hilbert spaces H_i, $i = 1, 2$ are equivalent ($U_1 \cong U_2$) if there exists an isometry T of H_1 onto H_2 such that $TU_1(g) = U_2(g)T$, g in G. We consider the following structure on the set of equivalence classes of unitary representations of G which we denote by $R(G)$.

(1) The direct sum $U_1 \oplus U_2$ of two representations U_i on H_i is the representation of G on $H_1 \oplus H_2$ given by $U_1 \oplus U_2(g) \langle x_1, x_2 \rangle = \langle U_1(g)x_1, U_2(g)x_2 \rangle$, where $\langle x_1, x_2 \rangle$ is in $H_1 \oplus H_2$, and g is in G. This addition is well-defined on $R(G)$ and is commutative and associative.

A generalization of direct sum, essential for the case where G is not abelian or compact, is that of the continuous direct sum often called the direct integral. Let (Y, S, μ) be a measure space with S the class of measurable sets on Y and μ the measure, and let $\{H_y\}$, y in Y be a collection of Hilbert spaces. A direct integral of $\{H_y\}$, y in Y is a subspace H of functions f of Y into $\bigcup_{y \in Y} H_y$ having the properties (i) if f is in H then $f(y)$ is in H_y; (ii) there exists a countable family $f_n \in H$ such that for almost all $y \in Y$, $\{f_n(y)\}$ spans H_y; and H is maximal with respect to (iii) if f and g are in H, then $(f(y), g(y))_y$ is a measurable function on (Y, S, μ) and lies in $L_1(Y, S, \mu)$. Here $(\cdot, \cdot)_y$ denotes the inner product in

H_y. H is a Hilbert space with inner product given by

$$(f, g) = \int_Y (f(y), g(y))_y \, d\mu.$$

If to each point y in Y we make correspond a bounded operator ${}^yT$ on H_y in such a manner that $({}^yTf(y), g(y))_y$ is measurable and ess sup $\| {}^yT \|_y < \infty$, then the collection $\{{}^yT\}$ defines a bounded operator T on H, where $(Tf)(y) = {}^yT(f(y))$. T will be denoted by $\int {}^yT \, d\mu$. Suppose now that $\{{}^yU\}$ is a set of unitary representations of G on H_y, y in Y for which $\int {}^yU(g) \, d\mu$ is defined for each g in G. Then $U = \int {}^yU \, d\mu$ is the unitary representation of G on $\int H_y \, d\mu$ given by

$$(U(g)f)(y) = \int {}^yU(g)(f(y)) \, d\mu.$$

For a detailed discussion of many technical questions here omitted in the definition of continuous direct sum, see [37, 66, 87, 101].

(2) If U is a representation of G on H, the adjoint representation of U is a representation $\bar{U}$ on $\bar{H}$, the dual space of H given by $\bar{U}(g)\varphi = U(g^{-1})^*\varphi$, φ in $\bar{H}$, * denoting the usual adjoint mapping from operators on H to operators on $\bar{H}$. It is easy to check that if $U_1 \cong U_2$, then $\bar{U}_1 \cong \bar{U}_2$ and

$$\overline{\int {}^yU \, d\mu} = \int {}^y\bar{U} \, d\mu.$$

(3) Let $H_1 \otimes H_2$ be the set of all linear transformations T from $\bar{H}_2$ to H_1 such that $\alpha_2 T^* \alpha_1 T$ is a Hilbert-Schmidt operator of $\bar{H}_2$ into $\bar{H}_2$ where α_i is the conjugate linear map of H_i onto $\bar{H}_i$. The trace on such operators makes $H_1 \otimes H_2$ into a Hilbert space, the Kronecker product of H_1 and H_2. The Kronecker product of representations U_1 and U_2 is a representation $U_1 \otimes U_2$ of G on $H_1 \otimes H_2$ given by $(U_1 \otimes U_2)(g)T = U_1(g)T\bar{U}_2(g)$, g in G, T in $H_1 \otimes H_2$. On $R(G)$, the Kronecker product is commutative and associative. Also, $\overline{U_1 \otimes U_2} \cong \bar{U}_1 \otimes \bar{U}_2$ and

$$U \otimes \int {}^yU \, d\mu \cong \int U \otimes {}^yU \, d\mu.$$

The set of self-adjoint operators in $H \otimes H$ is a subspace $H \circledS H$ left invariant by $U \otimes U$, as is its orthogonal complement $H \circledA H$, the set of skew adjoint operators. The restriction of $U \otimes U$ to $H \circledS H$ we denote by $U \circledS U$ and call the symmetric Kronecker square. The anti symmetric Kronecker square $U \circledA U$ is the restriction of $U \otimes U$ to $H \circledA H$. Clearly $U \otimes U \cong (U \circledS U) \oplus (U \circledA U)$. What has just been defined for the symmetric group on two letters S_2 acting on $H \otimes H$ can be done more generally for S_n acting on $H \otimes \cdots \otimes H$, the product taken n times, by permuting the factors. This will give a decomposition of $U \otimes \cdots \otimes U$ into a direct sum, each summand corresponding to a partition of n.