Pseudo-differential Operators and the Nash–Moser Theorem

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Pseudo-differential Operators and the Nash–Moser Theorem

Preface to the English edition

We are happy to welcome the English translation of our book, which originally appeared under the title 'Opérateurs pseudo-différentiels et théorème de Nash–Moser' in 1991 (InterEditions/Editions du CNRS, Paris).

Though the world of partial differential equations has changed a lot during these years, we think that the elementary presentation of the subjects touched upon in our book is still up to date and can be useful; thus, we made no changes, except for correcting some misprints. On the other hand, several remarkable books on partial differential equations have appeared since: though their scopes largely exceed that of our book, we thought it relevant to mention them in our bibliography.

Finally, we wish to thank the translator, Dr. Stephen S. Wilson, and the editorial board of the AMS, who worked to produce this new edition of our work.

Orsay, November 2006 — Serge Alinhac and Patrick Gérard

Translator's note

The numbering system I have used in my translation is essentially that employed by the authors in the original French edition so that the actual equation numbers etc. are the same in both versions. I did, however, make certain changes to the cross-referencing system: for example, to remove ambiguity, outside of Chapter II Exercise A.1 of that chapter may be referred to here as Exercise II.A.1 although within Chapter II it is referred to as Exercise A.1.

Cheltenham, February 2007 — Stephen S. Wilson

Contents

General introduction

This book provides an elementary exposition intended for students who have completed four years' study of university level mathematics. A knowledge of the elements of functional analysis, Fourier analysis and distribution theory (including, in particular, Fourier analysis in $\mathcal{S}$ and $\mathcal{S}'$) is assumed. Chapter 0 contains a reminder of the notation, concepts and main results used in the remainder of the book (with references). On the other hand, no knowledge of partial differential equations is needed, although it will be beneficial to have received an initiation to the topic.

The book stems from a course on 'Pseudo-differential operators and the Nash–Moser theorem', presented at the Ecole Normale Supérieure (ENS) from October 1986 onwards, to second-year students studying for the degree of Master of Fundamental and Applied Mathematics and Computer Science.

Although the topics covered largely form the subject of research literature, we have striven to avoid any scholarly discussions, 'veiled references' and sibylline allusions, which might open chasms beneath the reader's footsteps. A particular presentation of the subject is selected and developed in each chapter: the commentary at the end of each chapter indicates the sources, differing approaches, certain current extensions, and the connections between the topics handled.

Finally, we have assembled numerous exercises, divided into two classes. Elementary exercises are intended to help readers assimilate the course and monitor their progress. Other more complex exercises, marked with an asterisk ($*$), present recent developments which have sometimes only been published in journal articles: we crave their authors' forgiveness for this simplification! These exercises, unlike those in certain famous treatises, can be effectively solved by real students, as experience of the teaching at ENS has shown.

It was our wish that this text should also be useful to researchers as a simple and *self-contained* introduction to subjects with which they are unfamiliar.

The dual purpose of these notes led us to keep them short, sometimes at the expense of a certain denseness of the text (which we believe is essentially accessible to motivated students). In particular, we had in mind our many colleagues in 'applied mathematics' who wish to use the Nash–Moser theorem in their research or to keep themselves up to date on microlocal analysis, without delving into the arcana of the specialist literature: they will be able to read the desired chapters independently of each other.

The choice of the material presented is a matter of personal taste and of the fields of research of the authors who, incidentally, believe that certain difficult (nonlinear) problems cannot be solved without a sufficient knowledge of pseudo-differential operators.

The authors are indebted to numerous mathematicians (cited in the commentaries) who have inspired them to present the subjects dealt with, and, in particular, to L. Hörmander, to whom the mathematical contents of Chapter I and Section III.C are largely due. The Bibliography at the end of the book indicates the sources used.

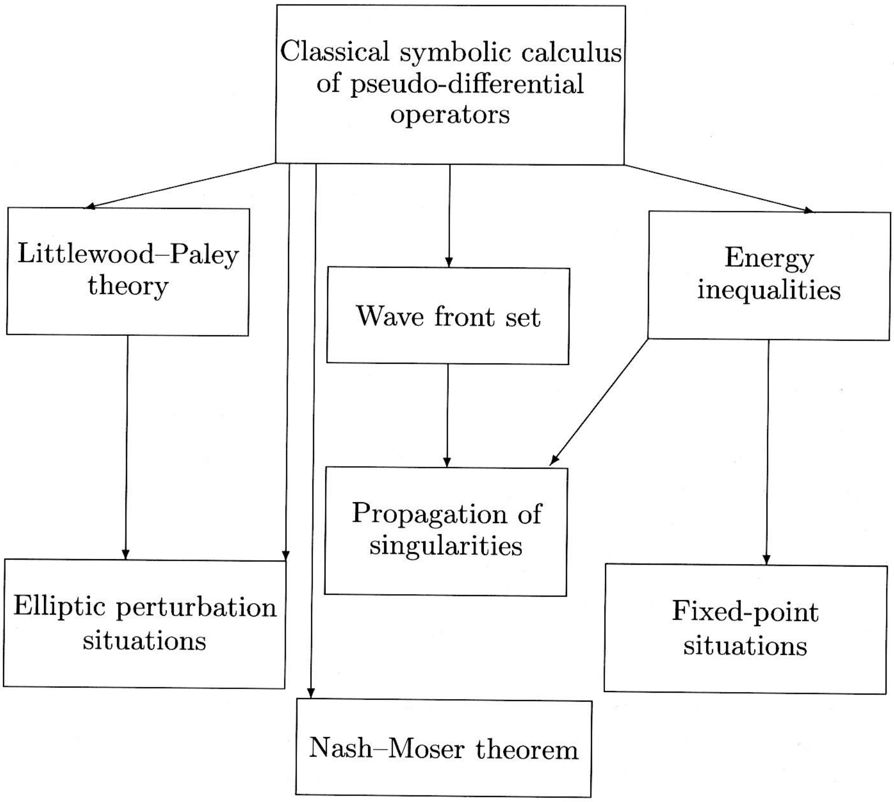
While presenting important concepts which are the true starting points for numerous recent developments, we have sought to end up with real theorems: microlocal elliptic regularity; propagation of singularities; existence of solutions of quasilinear hyperbolic systems; existence of isometric embeddings; the Nash–Moser theorem. The plan of the book is as follows.

In Chapter I we present the 'minimal' theory of pseudo-differential operators, in a global context (on $\mathbb{R}^n$), which turns out to be very nice in practice. The main points here are the notion of the symbol, the symbolic calculus for operators, the action in Sobolev spaces and the invariance under change of coordinates. The text presents only a few concrete applications and the most technical proofs are brought together in the appendix, in order to enable the reader to obtain an overall view of the subject. The exercises in Chapter I, which are particularly numerous, provide an introduction to a number of variants of the theory proposed and present several applications, notably to the analysis on compact manifolds.

Chapter II brings together three themes. Section A presents the Littlewood–Paley theory of 'dyadic decomposition' of distributions: this systematizes the natural division of the space of frequencies ξ according to their size $|\xi|$, associated with the classical symbolic calculus of Chapter I. This very simple theory allows one to rapidly obtain interesting properties of composite functions in Sobolev and Hölder spaces. Section B presents the concept of the wave front set and its links with pseudo-differential operators: this time it is a matter of the conical division of the space of frequencies ξ according to their directions $\xi \in S^{n-1}$, associated with the classical symbol homogeneities. Finally, Section C deals with hyperbolic energy inequalities for which pseudo-differential operators turn out to be an effective tool. Thus, Chapter II serves to present very useful applications of the 'dry theory' of Chapter I, while preparing the material and the concepts which will be needed in Chapter III.

The final chapter discusses certain problems of a nonlinear nature which arise in geometry or in analysis and which may be reduced to perturbation problems. The plan of this chapter reflects the various situations which one may encounter: 'elliptic' situations in which the usual Banach implicit function theorem suffices; 'fixed-point' situations, such as one often finds in nonlinear hyperbolic problems or again in the isometric embedding problem; and, finally, situations where the 'loss of derivatives' is too great and a Nash–Moser technique has to be used. The Nash–Moser theorem relies completely on the acquisition of *a priori* 'tame' inequalities; the reader who is already familiar with *a priori* inequalities (presented in Chapter I and Section III.C) will grasp the concept of 'tame' estimates through its clear link with Littlewood–Paley theory and the paradifferential calculus of J.-M. Bony (Section II.A).

This establishes the underlying cohesion of this book, which can be schematized in the accompanying diagram.



In this spirit, we were recently very happy to learn of the work of L. Hörmander [H9], explaining the links between pseudo-differential and paradifferential operators, fixed-point methods and the Nash–Moser theorem.

Finally, we are grateful to G. Ben Arous and J.B. Bost for their kind and valuable suggestions.

Chapter 0

Notation and review of distribution theory

In this chapter, we introduce the various notation used in the book, while recalling a number of elements of distribution theory and Fourier analysis which will be used throughout. In the following chapters, we shall thus assume that the reader is familiar with these notions. Nevertheless, a less advanced student will be able to find the results cited below in the book by J. Chazarain and A. Piriou [CP] (Chapter 1, Sections 1, 2 and 4) or in that by W. Rudin [R]. Readers with little knowledge of distributions are advised to read the book by L. Schwartz [S] beforehand, while students who wish to test their knowledge in this area will find a large number of exercises with solutions, together with review material in the work of C. Zuily [Z].

1. Spaces of differentiable functions and differential operators

Let Ω be an open subset of $\mathbb{R}^n$. If k is a nonnegative integer, then we let $C^k(\Omega)$ denote the space of k-times continuously differentiable functions on Ω with values in $\mathbb{C}$. Similarly, $C^\infty(\Omega)$ denotes the space of indefinitely differentiable functions on Ω. This notation extends first to the case where $\Omega = M$ is a differentiable manifold and, second, to the case where the co-domain is not $\mathbb{C}$ but a topological vector space E on $\mathbb{R}$: we then denote the corresponding spaces by $C^k(\Omega, E)$ and $C^\infty(\Omega, E)$.

For $k \in \mathbb{N} \cup \{\infty\}$, $C_0^k(\Omega)$ denotes the subspace of $C^k(\Omega)$ whose elements are zero outside a compact subset of Ω, while $C^k(\overline{\Omega})$ is formed by the restrictions to Ω of elements of $C^k(\mathbb{R}^n)$.

We shall use *multiple indices* to denote the partial derivatives of an element of $C^k(\Omega)$. A multiple index $\alpha = (\alpha_1, \dots, \alpha_n)$ is an element of $\mathbb{N}^n$, its *modulus* $|\alpha|$ is by definition $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and we set $\alpha! = \alpha_1! \dots \alpha_n!$. For $j \in \{1, \dots, n\}$, the derivative $\frac{\partial}{\partial x_j}$ will also be denoted by ∂_{x_j} or ∂_j when there is no risk of confusion. For reasons associated with the Fourier transformation (see Section 5 below), it is also useful to introduce the notation $D_j = -\mathrm{i}\frac{\partial}{\partial x_j}$. A higher-order derivative will then be denoted by $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ or $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$. We shall also use this convention to denote the monomials constructed in the components of a vector of $\mathbb{R}^n$. Thus, if $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

A *differential operator* on Ω is a finite linear combination of derivatives of arbitrary orders with coefficients in $C^\infty(\Omega)$. It is said to be *of order* m if it does not include derivatives of order greater than m. In other words, a differential operator of order m on Ω can be written as

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

where the $a_\alpha \in C^\infty(\Omega)$ are the coefficients of P. In this form, it is easy to see that P defines a linear mapping from $C^{k+m}(\Omega)$ to $C^k(\Omega)$ for all k. The *symbol* P is the polynomial function in ξ defined on $\Omega \times \mathbb{R}^n$ by

$$p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha,$$

while its *principal symbol of order* m (or *principal symbol* if there is no risk of confusion) is the homogeneous function in ξ

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

2. Distributions on an open set of $\mathbb{R}^n$

a) A *distribution* on an open set Ω is a linear form u on $C_0^\infty(\Omega)$ satisfying the following continuity property: for any compact subset K, there exist an integer m and a constant C such that for all $\varphi \in C_0^\infty(\Omega)$ which are zero outside K,

$$|\langle u, \varphi \rangle| \leq C \sup_{x \in K} \sup_{|\alpha| \leq m} |\partial^\alpha \varphi(x)|.$$

The space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. In particular, it contains the space $L^1_{\mathrm{loc}}(\Omega)$ of *locally integrable* functions on Ω, via the following identification:

(2.1) $$\forall \varphi \in C_0^\infty(\Omega),\ \forall f \in L^1_{\mathrm{loc}}(\Omega), \quad \langle f, \varphi \rangle = \int f(x) \varphi(x) \mathrm{d}x.$$

Another example of a distribution is given by the *Dirac mass* at a point. If $x_0 \in \Omega$ and $\varphi \in C_0^\infty(\Omega)$, we denote $\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0)$.

b) Let $u \in \mathcal{D}'(\Omega)$. Define $\partial_j u \in \mathcal{D}'(\Omega)$ by the formula

$$\langle \partial_j u, \varphi \rangle = -\langle u, \partial_j \varphi \rangle,$$

which, taking into account the identity (2.1), extends the operator ∂_j previously defined on $C^1(\Omega)$ to distributions. Similarly, if $a \in C^\infty(\Omega)$, $au \in \mathcal{D}'(\Omega)$ is defined by

$$\langle au, \varphi \rangle = \langle u, a\varphi \rangle.$$

Thus, any differential operator $P = \sum a_\alpha D^\alpha$ extends to a linear mapping from $\mathcal{D}'(\Omega)$ to itself, via the formula

$$\langle Pu, \varphi \rangle = \langle u, {}^tP\varphi \rangle,$$

where ${}^tP\varphi = \sum (-1)^{|\alpha|} D^\alpha (a_\alpha \varphi)$.

c) If Ω' is an open subset of Ω, and if $u \in \mathcal{D}'(\Omega)$, then the *restriction* $u|_{\Omega'}$ of u to Ω' is just the restriction of the linear form u to the space $C_0^\infty(\Omega') \subset C_0^\infty(\Omega)$. Then u is said to be zero (resp. of class C^k) on Ω' if $u|_{\Omega'} = 0$ (resp. $u|_{\Omega'}$ can be defined by $f \in C^k(\Omega')$, according to formula (2.1)). For this definition to be manageable, it is important that one should be able to recover u from its restrictions to the open sets of a covering of Ω. This is the object of the following lemma.

Lemma (Partitions of Unity). *Let (Ω_j) be a family of open subsets of Ω such that $\Omega = \bigcup_j \Omega_j$. Then there exists a family of functions (φ_j) such that:*

i) $\forall j, \varphi_j \in C^\infty(\Omega)$, $\operatorname{supp}(\varphi_j) \subset \Omega_j, 0 \le \varphi_j \le 1$.

ii) *For any compact subset K of Ω, $\{j, K \cap \operatorname{supp} \varphi_j \neq \emptyset\}$ is finite.*

iii) *In Ω, $\sum_j \varphi_j = 1$. (This sum is well defined, following* ii)*).*

In addition to the references already cited, the reader may refer to Exercise 6.1 of Chapter I for a proof of the above lemma (given under the hypothesis that $\overline{\Omega}_j$ is compact in Ω; the general case is an easy consequence).

Using this lemma, one can show, for example, that if $\bigcup_j \Omega_j = \Omega$ and if $u|_{\Omega_j} = 0$ (resp. $u|_{\Omega_j} \in C^k$) for all j, then $u = 0$ (resp. $u \in C^k(\Omega)$).

This leads us to the following definitions: the *support* of u (resp. *singular support* of u) is defined to be the complement in Ω of the points in the neighbourhood of which u is zero (resp. u is of class C^∞). The support of u is denoted by $\operatorname{supp} u$; the singular support of u is denoted by sing supp u. These are two closed sets, satisfying sing supp $u \subset \operatorname{supp} u$, and the preceding result can be paraphrased by the equivalences

$$u = 0 \Leftrightarrow \operatorname{supp} u = \emptyset,$$
$$u \in C^\infty \Leftrightarrow \text{sing supp } u = \emptyset.$$

Finally, we note that if $u \in C^0(\Omega)$, then the support of u defined above coincides with the *closure* of $\{x \in \Omega, u(x) \neq 0\}$.

The space of distributions *with compact support* in Ω is denoted by $\mathcal{E}'(\Omega)$. It is identified with the space of linear forms on $C^\infty(\Omega)$ which are continuous for the topology defined by the semi-norms

$$\sup_{x\in K} \sup_{|\alpha|\leq m} |\partial^\alpha \varphi(x)|,$$

where K runs over the compact subsets of Ω and m runs over the integers.

3. Convolution

a) Let u and v be two C^∞ functions with compact support. We set

$$u * v(x) = \int u(y)v(x-y)\mathrm{d}y = \int u(x-y)v(y)\mathrm{d}y. \tag{3.1}$$

The function $u * v$ thus defined is C^∞ with compact support and satisfies

$$\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v. \tag{3.2}$$

This is called the *convolution* of the two functions u and v.

Of course one can define the convolution of less regular functions. The most natural extension relates to summable functions: if u and v belong to $L^1(\mathbb{R}^n)$, then $u * v$ defined by (3.1) belongs to $L^1(\mathbb{R}^n)$ and we have

$$\int |u * v(x)|\mathrm{d}x \leq \int |u(x)|\mathrm{d}x \cdot \int |v(x)|\mathrm{d}x.$$

However, it is not this extension which we shall use the most frequently, but rather that described in sections b) and d) below, which relate to the cases where $u \in \mathcal{D}'(\mathbb{R}^n)$, $v \in C_0^\infty(\mathbb{R}^n)$, and then where $u \in \mathcal{D}'(\mathbb{R}^n)$, $v \in \mathcal{E}'(\mathbb{R}^n)$.

b) Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in C_0^\infty(\mathbb{R}^n)$; then the formula

$$u * v(x) = \langle u, v_x\rangle, \text{ with } v_x(y) = v(x-y),$$

defines a function $u * v$ of class C^∞ on $\mathbb{R}^n$. This function also satisfies

$$\partial^\alpha(u * v) = \partial^\alpha u * v = u * \partial^\alpha v, \tag{3.3}$$

$$\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v. \tag{3.4}$$

c) The convolution is the basis for a very useful *regularization* procedure which we now describe.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be nonnegative with integral equal to 1, and let $\varepsilon > 0$; we set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then if $u \in \mathcal{D}'(\mathbb{R}^n)$, the family of C^∞ functions

$u_\varepsilon = u * \varphi_\varepsilon$ converges to u as ε tends to 0, in the sense that

(3.5) $$\forall \psi \in C_0^\infty, \int u_\varepsilon(x)\psi(x)\mathrm{d}x \to \langle u, \psi \rangle.$$

This procedure of approximation by regular functions is of interest because the mode of convergence of u_ε to u is essentially described by the regularity of u. Thus, if $u \in C^k(\mathbb{R}^n)$, u_ε converges to u in the sense of the semi-norms $\sup_{x\in K} \sup_{|\alpha|\le k} |\partial^\alpha v(x)|$, where K runs over the compact subsets of $\mathbb{R}^n$; if $u \in L^p(\mathbb{R}^n)$ $(1 \le p < +\infty)$, the space of functions summable to the pth power, then u_ε tends to u in L^p.

Moreover, equation (3.4) shows that the support of u_ε is arbitrarily close to that of u when ε tends to 0. Using a 'cut-off function', we thus extend the regularization procedure to distributions defined on an open set Ω of $\mathbb{R}^n$, showing, for example, that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ if $p \in [1, +\infty[$ and in $\mathcal{D}'(\Omega)$ for the 'weak topology', i.e. in the sense of (3.5).

d) To define the convolution of two distributions, we first observe that if $u \in \mathcal{D}'(\mathbb{R}^n)$, $v, \varphi, \in C_0^\infty(\mathbb{R}^n)$, then

$$\int u * v(x)\varphi(x)\mathrm{d}x = \langle u, \tilde{v} * \varphi \rangle,$$

where we have set $\tilde{v}(x) = v(-x)$.

After having extended the operator $v \mapsto \tilde{v}$ to distributions by

$$\langle \tilde{v}, \varphi \rangle = \langle v, \tilde{\varphi} \rangle,$$

we set, for $u \in \mathcal{D}'(\mathbb{R}^n)$, $v \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\langle u * v, \varphi \rangle = \langle u, \tilde{v} * \varphi \rangle.$$

We thus define a distribution $u * v$ on $\mathbb{R}^n$, which again satisfies (3.3) and (3.5), to which can be added

(3.6) $$\text{sing supp } (u * v) \subset \text{sing supp } u + \text{sing supp } v.$$

For example, if $\delta = \delta_0$ denotes the Dirac mass at the origin, then for any distribution u on $\mathbb{R}^n$, we have $u * \delta = u$. The convolution of distributions is fundamental in the study of differential operators with constant coefficients. An illustration of this can be found in Section I.1.1, where the proof of the relation (3.6) is also outlined.

4. Kernels

Let Ω_1 and Ω_2 be two open subsets of $\mathbb{R}^n$, and let $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. The equation

(4.1) $$\langle A_K v, u \rangle = \langle K, u \otimes v \rangle,$$

where $u \in C_0^\infty(\Omega_1)$, $v \in C_0^\infty(\Omega_2)$, $u \otimes v(x_1, x_2) = u(x_1)v(x_2)$, defines a linear mapping $A_K : C_0^\infty(\Omega_2) \to \mathcal{D}'(\Omega_1)$, which is continuous in the following sense: for all u in $C_0^\infty(\Omega_1)$, for all compact subsets K of Ω_2, there exist a constant C and an integer m such that, $\forall v \in C_0^\infty(\Omega_2)$ with support in K,

$$|\langle A_K v, u\rangle| \le C \sup_{x \in K} \sup_{|\alpha| \le m} |\partial^\alpha v(x)|. \tag{4.2}$$

When $K \in L^1_{\mathrm{loc}}(\Omega_1 \times \Omega_2)$, equation (4.1) can be written in the more familiar form

$$A_K v(x_1) = \int K(x_1, x_2) v(x_2) \mathrm{d}x_2.$$

In general, the distribution K is entirely determined by equation (4.1) and is called the *kernel* of the operator A_K.

A theorem due to L. Schwartz ensures that any operator $A : C_0^\infty(\Omega_2) \to \mathcal{D}'(\Omega_1)$, which is continuous in the sense of (4.2), has a kernel. However, we shall never use this theorem in what follows, since the operators which we shall manipulate have readily identifiable kernels.

For example, the kernel of the differential operator $P = \sum a_\alpha(x) D^\alpha$ is the distribution $K(x_1, x_2) = \sum a_\alpha(x_1) D^\alpha \delta(x_1 - x_2)$, where δ is the Dirac mass; the kernel of the operator of convolution by the distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is $K(x_1, x_2) = u(x_1 - x_2)$.

The notion of kernel thus gives rise to a more algebraic study of operators. For example, given an operator A with kernel K, it is easy to define the *transpose* operator for A (denoted by ${}^tA$), characterized by

$$\forall u \in C_0^\infty,\ \forall v \in C_0^\infty, \quad \langle Av, u\rangle = \langle {}^tAu, v\rangle.$$

It suffices to take ${}^tA$ to be the operator with kernel

$${}^tK(x_1, x_2) = K(x_2, x_1).$$

The kernel also enables us to control the supports (and the singular supports). For example,

$$\operatorname{supp}(A_K v) \subset \{x_1, \exists x_2 \in \operatorname{supp} v, (x_1, x_2) \in \operatorname{supp} K\}.$$

A similar relation exists for the singular support.

5. Fourier analysis on $\mathbb{R}^n$

a) We first introduce the space $\mathcal{S}$ of C^∞ functions which decrease rapidly on $\mathbb{R}^n$. This is the space of C^∞ functions u on $\mathbb{R}^n$, satisfying

$$\forall \alpha \in \mathbb{N}^n,\ \forall \beta \in \mathbb{N}^n, \quad \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < +\infty.$$