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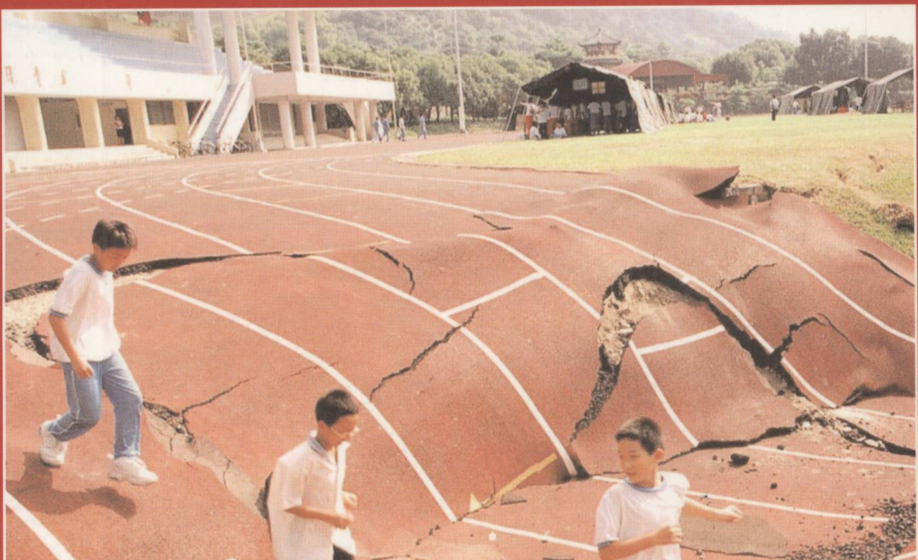

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Introduction

Whenever we see or hear anything, we do so because of the existence of waves. Electromagnetic waves cover a spectrum from low frequency radio waves, through visible light to X- and gamma rays. Sound propagates as a wave through the air. When someone sings or plays a musical instrument, the standing waves in their vocal chords, guitar strings or drumskins produce a pressure change, or sound wave, which is audible. Although these examples alone would be sufficient to motivate their study, wave phenomena occur in many other physical systems. Waves can propagate both on the surface of solid bodies (for example, as earthquakes) and through the bulk of a solid (for example, in seismic oil prospecting). The surface of the sea is perhaps the most obvious example of a wave bearing medium. Water waves vary in size from the small ripples caused by raindrops, through shock waves such as the Severn bore, to enormous ocean waves that can capsize large ships. Waves in different media can interact, often with devastating effects, for example when an underwater earthquake causes a tsunami, a huge wall of water that can destroy coastal settlements, or when the waves generated by the wind blowing on a bridge produce a catastrophic resonance.

Wave phenomena emerge in unexpected contexts. The flow of traffic along a road can support a variety of wave-like disturbances as anybody who has sat in slowly moving traffic will know. The beat of your heart is regulated by spiral waves of chemical activity that swirl across its surface. You control the movement of your body through the action of electrochemical waves in your nervous system. Finally, quantum physics has revealed that, on a small enough scale, everything around us can only be described in terms of waves.

A question that we might reasonably ask at this point is 'what is a wave?' Well, the first place you might look is a dictionary, where waves are usually defined as 'disturbances propagating in water at a finite speed'. However, this is not satisfactory since standing waves do not propagate and wave propagation is possible in media other than water. Another feature that makes definitions difficult is the interaction of the wave with the medium through which it is passing. A wave on the surface of a pond passes by and leaves the medium unchanged. In contrast, a chemical wave usually leaves the reacting species involved in a different chemical state after it has passed by. For these reasons we believe that there is no single definition of waves, and choose to view them as a generic set of phenomena with many similarities.

To cover all of these topics would be an enormous task. We have written this textbook with the needs of advanced undergraduates in applied mathematics in mind, and have restricted the range of material covered accordingly. We hope that the book will also be of use to physics and engineering students. It is worth noting that this is not a book about numerical methods for wave propagation. This reflects both the need to keep the book to a manageable size and the interests of the authors. The emphasis is on analytical and asymptotic methods for the investigation of systems of equations with wave-like solutions. Much of the material is concerned with various aspects of fluid mechanics, but we also cover basic elastic, electromagnetic, traffic, chemical and electrochemical waves. We assume that the reader has taken basic courses in fluid mechanics, elasticity (for chapter 5), partial differential equations, vector calculus, phase plane methods (for chapter 9) and asymptotic methods. Throughout we have tried to emphasise the mathematical similarities between the disparate areas that we deal with, in terms of both the structure of the governing equations and the techniques available for their solution. At the end of each chapter there are exercises, designed to both reinforce and augment the material presented in the chapter. Bona fide teachers and instructors may obtain full worked solutions to many of these by emailing dtranah@cup.cam.ac.uk.

The book is divided into three parts. Linear waves are dealt with in part one. All of the analytical techniques of nineteenth century mathematics can be brought to bear on linear wave equations. The techniques of separation of variables, Fourier series and Fourier transforms are used to reveal the properties of linear waves on stretched strings (chapter 2), linear sound waves (chapter 3), linear water waves (chapter 4), linear waves in solids (chapter 5) and electromagnetic waves (chapter 6).

In part two we cross the threshold into the twentieth century and study nonlinear waves. We begin in chapter 7 by examining hyperbolic systems governed by the propagation of information on characteristics. As examples we use traffic flow and nonlinear gas dynamics. We then move on to study nonlinear water waves (chapter 8), an extension of the material presented in chapter 4, and then chemical and electrochemical waves (chapter 9).

The third and final part of the book covers more advanced topics. In chapter 10 we consider various physical systems that can be modelled using Burgers' equation. These include a more sophisticated model for the flow of traffic, and weakly nonlinear compressible gas dynamics. Chapter 11 is concerned with the analysis of scattering and diffraction of both scalar and vector waves, through apertures and past obstacles. In chapter 12 we describe the use of the inverse scattering transform to solve the Korteweg–de Vries equation (KdV) and the nonlinear Schrödinger equation (NLS). The KdV equation governs, amongst other things, the propagation of long waves on shallow water, whilst the NLS equation governs the propagation of dispersive wavepackets in a nonlinear medium, for example pulses of light in optical fibres. This allows us to illustrate some of the remarkable properties of solitons. These are localised waves that can retain their identity after nonlinear interactions.

Throughout the first two parts of the book, more advanced topics that do not fit naturally into part three, and which could be omitted at first reading, have been marked with an asterisk.

In writing this book we have benefited from the advice and encouragement of many colleagues. Particular gratitude is due to David Crighton for the invitation to write a book in this series. He also taught a stimulating course on waves to both authors as undergraduates. The various chapters of this book have been read and commented upon prior to publication by John Blake, Stephen Decent, Eammon Gaffney, Yulii Shikhmurzaev, Athanasios Yannacopoulos and Ray Jones (all at Birmingham), Tony Rawlins (Brunel), Peter Smith (Keele), Nigel Scott and Jean-Marc Vanden-Broeck (East Anglia), Howell Peregrine (Bristol), Sam Falle (Leeds), David Parker (Edinburgh) and Colin Pask and Rowland Sammut (New South Wales). Any remaining mistakes are, of course, our own. We also acknowledge and thank the copyright owners of the many photographs and drawings throughout the text: Lawrence Coates, Ken Elliott and Graham Westbrook (Birmingham), Neville Fletcher (ANU), Howell Peregrine (Bristol), Tim Rees (Transport Research Laboratory Ltd), Malcolm Bloor (Leeds), Geraint Thomas (Aberystwyth), Bernard Richardson (Cardiff), Martin Boeckmann (Magdeburg), Germany Aerospace Centre, Berlin, BAE Systems

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ACK and JB, Birmingham 2000

Part one

Linear Waves

1

Basic Ideas

In this chapter, we introduce some of the generic ideas and notation that underpin the analysis of waves in the physical, chemical and biological systems that we will study in subsequent chapters. We consider some typical, linear partial differential equations with wave-like solutions, and discuss some of their qualitative and quantitative features. We also present the method of stationary phase, which gives us a straightforward way of determining how the waves in a linear system behave a long time after they are generated.

A partial differential equation is **linear** if all of the dependent variables appear linearly. Consider, for example, the one-dimensional wave equation,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}. \tag{1.1}$$

This governs, amongst other things, the propagation of small amplitude waves on a stretched string, as we shall see in chapter 2. Here ϕ is the amplitude of the displacement of the string, x position along the string and t time. This is a linear equation, since $\phi(x,t)$ only appears linearly (there are no **nonlinear** terms, such as ϕ^2 or $\phi\partial\phi/\partial x$).

The most important feature of solutions of linear equations is that there is a **principle of superposition**. If $\phi = \phi_1$ and $\phi = \phi_2$ are solutions of a linear equation, then $a_1\phi_1 + a_2\phi_2$ is also a solution for arbitrary constants a_1 and a_2. In addition, if $\phi(x,t;K)$ is a solution for any value of the constant K then

$$\int_{-\infty}^{\infty} \phi(x,t;K)\,dK$$

is also a solution. This feature allows us to build up the solution of a