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Nonlinear evolution equations

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Nonlinear evolution equations

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Abstract

This work reflects contemporary understanding in the modelling of nonlinear wave processes in weakly dispersive media. The fruitful notion of evolution equations governing the propagation of single waves is used. Three methods are described for the construction of nonlinear evolution equations: the iterative, the asymptotic and the spectral. A comparative analysis of these methods is presented including the problems of the convergence and correctness. Several physical situations are discussed involving one-dimensional, weakly inhomogeneous and/or wave-guide systems. The simple evolution equations are analysed separately.

This book has been written from the viewpoint of graduate students in wave mechanics and mathematical physics and may also be of interest to postgraduate students and research workers in those branches of mathematical physics and engineering that are concerned with nonlinear wave propagation.

Preface to the English edition

Mathematical physics requires good definitions on which mathematical studies can be based. To us, one of the important notions in wave theory is the concept of single waves. Starting from this point, we return to the parts of mathematics in which the notion of evolution equations has played a significant role in the contemporary understanding on nonlinear waves. These research notes describe the methods available for constructing the evolution equations governing nonlinear wave propagation. After introducing the possible methods together with their comparative analyses, these methods are then applied to various complicated physical situations. The cases considered include wave-beams, the description of near-caustic zones, the elimination of "non-wave"coordinates, etc. The notes are written mainly on the basis of the research carried on by the authors themselves. Their scientific interests are related to nonlinear wave propagation, with applications in hydrodynamics, acoustics, mechanics, biophysics, etc. The material given in the notes has also been used for teaching at both the graduate and post-graduate level.

The authors, whose names are listed in the alphabetical order, share equal responsibility for these notes.

The translation of the notes was finished when one of the authors (JKE) was at the University of Newcastle upon Tyne, U.K., supported by a grant from the SERC. The authors are very much indebted to the SERC and would like to thank Professor Alan Jeffrey for the help in improving the presentation of the notes.

JKE, VEF, ENP

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Introduction

Wave propagation theory is historically most closely related to the development of approximate methods for solving partial differential and/or integro-differential equations (systems of equations). However, even in the linear approach when finite deformations are neglected, the number of exact solutions describing the dependence of the field variables on the initial conditions is rather small. These cases, especially when three-dimensional diffraction problems and waves in dispersive media are considered, represent nowadays the classical examples of mathematical physics. The number of exactly solvable nonlinear problems is certainly much smaller. This is the main reason why approximate methods are so intensely developed for both linear and nonlinear problems. Generally speaking, the approximate methods used in the wave propagation theory may be divided into three main groups:

i) the approximate analysis of the exact solution;

ii) the perturbative analysis of the solution with small (slow) derivation from a known one;

iii) the simplification of mathematical models (equations) describing the process.

The methods of group (i) take the exact solution written in the integral form as a basis, and the approximate solutions are obtained by means of some classical approximation method, for example, by means of the methods of stationary phase and steepest descent [114, 116]. The method of stationary phase, first proposed by Kelvin for solving the wave pattern formation behind a moving ship is now considered to be a classical one. A great many linear wave propagation problems may be solved nowadays by means of such an approach when the integral solutions are known. Unfortunately, it is practically impossible to get explicit integral formulae for nonlinear problems and this is a great drawback of this physically well grounded method.

The methods of the group (ii) are better in this sense because nonlinear problems are also solvable. The basic solutions are usually taken in the form of stationary plane waves. First of all, the traditional perturbation methods belong to this group. The straightforward perturbation procedure involves a series expansion with respect to a small parameter ε, where the first term is the solution of the problem with $\varepsilon = 0$ [73, 112]. The solution of the "reduced" problem ($\varepsilon = 0$) may be found either exactly or approximately. The instability of nonlinear waves is usually analysed according to such an approach. However, as a rule, the straightforward perturbation procedures break down in the course of time t (or the space coordinate) due to secular terms which make the solution unbounded as t tends to infinity. These difficulties can be avoided if the constant parameters of the basic solution, such as the amplitude and the frequency, for example, are considered to be slowly changing in time and in space. This permits the construction of uniformly valid approximate series (or their finite sums) representing the solution at a large time. The methods based on such procedures are called singular perturbation methods [46]. The reader can turn for details to various works on the topic [2, 20, 35, 44, 108, 116].

The methods of the group (iii) do not simplify the solutions, but rather the equations governing the wave process. At this stage no attention is paid to the solutions themselves. It is clear that the simplifying procedure ought to make use of certain small parameters which may either be present in the initial equations (systems of equations) or result from the process (the solution is close to the stationary one, for example). The wave process is thus described by the solution of the simplified equations [25]. The best results here are achieved when the initial system is simplified into a single equation, first order with respect to time and of arbitrary order with respect to space coordinates. This equation is called an *evolution equation*. Physically it means that the wave process is separated into single waves, each of them described by its own equation - an evolution equation. The best example of such an evolution equation is the well-known Korteweg-de Vries equation. Its derivation and history form a brilliant chapter in contemporary mathematical physics [62].

In this book we will be concerned with the methods for constructing non-linear evolution equations mainly in systems with the weak dispersion. Dispersion, as usual, means the dependence of the phase velocity on the frequency. In spite of weak dispersion the interaction between the different spectral components may be strong and the wave profiles may be of various shapes - from quasiharmonic waves up to N-waves and pulse-type waves. In this case it is rather difficult to fix a definite profile beforehand, therefore the most appropriate approach involves the simplification of the initial system without any concrete definition of the wave profile. The basic idea of simplification may be explained with the simple example of the linear wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} = 0. \qquad (0.1)$$

Its well-known general solution is represented by a super-position of two waves u and v:

$$U = u(x - ct) + v(x + ct). \qquad (0.2)$$

Each single wave, for example u, satisfies a simpler equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0. \qquad (0.3)$$

This is the simplest "single-wave" equation. Its solution coincides exactly with the solution of the initial equation (0.1) only when $v \equiv 0$, i.e. the wave process contains only one wave. The existence of a single wave is associated with certain initial and boundary conditions that actually may often occur in physics. This situation is in fact used in all the methods of simplification to a considerable extent. We shall demonstrate this approach on the physical level of strictness in connection with the model equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} - \varepsilon \frac{\partial^2 U^2}{\partial x^2} - 2\, \varepsilon \frac{\partial^4 U}{\partial x^4} = 0, \qquad (0.4)$$

where $\varepsilon \to 0$. Although the solution to the linearized equation (0.4) in the form of the sinusoidal progressive wavetrains is known [46], we shall not use it further. Provided $\varepsilon = 0$ the solution to the equation (0.4) is given by expression (0.2), and, if the single wave is realized then U is a function of one variable $y = x - ct$ only. In case of small ε's the single wave solution must be given in the form $U(x,t) = u(y,\tau = \varepsilon t)$. Substituting y and τ into the equation (0.4) we obtain

$$\varepsilon^2 \frac{\partial^2 u}{\partial \tau^2} - 2\varepsilon c \frac{\partial^2 u}{\partial \tau \partial y} - \varepsilon \frac{\partial^2 u^2}{\partial y^2} - 2\varepsilon \frac{\partial^4 u}{\partial y^4} = 0. \qquad (0.5)$$

The lowest-order variation leads to

$$c \frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial y} + \frac{\partial^3 u}{\partial y^3} = 0 \qquad (0.6)$$

which is the well-known Korteweg-de Vries equation.

As it is seen, the idea of simplification is rather simple itself, although its realization for complicated physical situations described by high-order equations (systems of equations) may be often difficult and cumbersome. In this connection a vast number of papers has been published in the sixties, dealing with the derivation of evolution equations in various physical situations. The mathematical side of these problems was often very similar, and as a matter of fact, the Korteweg-de Vries equations was often the main result of these investigations. The unified approach to constructing evolution equations of the high order was suggested by Taniuti and Wei [106] and actually marks a certain milestone in the theory. By making use of the fact that the initial system governing the wave propagation is close to the hyperbolic system the authors developed an asymptotic procedure in order to simplify the system. The lowest-order approximation of this procedure gave the evolution equations. Further investigations by the authors, their co-workers and others dealt with general expressions for the coefficients of the evolution equations, with the structure of the evolution equations depending on the structure of the initial system and higher approximations etc. Nowadays we have at our disposal a solid basis in order to understand the physical mechanism of single wave processes and further construction of evolution equations with the necessary exactness in many interesting problems of physics.

The existence of single waves is not the only condition used in methods of simplification. For quasi-one-dimensional waves (elastic waves in rods, surface waves etc.) the situation involving the transversal structure of the field (of a certain mode) is fixed in a large interval of frequency and/or wave numbers. In this case it is natural to eliminate the transversal (non-wave) coordinate along which the field structure is fixed. Such an approach permits a decrease in the number of independent variables. The best example here is the "classical" derivation of the equations governing the waves in shallow water [49]. We now give a brief derivation of this result. As is well known, the movement of an ideal liquid is governed by the equation

$$\Delta\varphi = 0 \quad (-H \leqq z \leqq \eta) \tag{0.7}$$

subject to the nonlinear boundary conditions at the free surface

$$\frac{\partial\eta}{\partial t} + \nabla\varphi\nabla\eta = \frac{\partial\varphi}{\partial z}, \tag{0.8}$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\eta = 0, \tag{0.9}$$

and at the bottom $(z = -H)$

$$\frac{\partial\varphi}{\partial z} = 0. \tag{0.10}$$

Here φ is the velocity potential, $\eta(x,y,t)$ is the displacement of the free surface, H is the depth of the basin, g is the acceleration of gravity and Δ is the Laplacian.

The difficulties in solving this problem are obvious. Meanwhile from the solution of the corresponding linear problem (see, for example, [54]) it is known that in the long-wave limit the solution is arranged comparatively simply: the pressure is hydrostatic, the field of horizontal velocities does not depend on the depth and the vertical velocity is considerably smaller than the horizontal velocity. This is true for waves with a wavelength considerably greater than the depth of the basin. Hence the depth may be considered as a small parameter. In this case the velocity potential may be expanded into Taylor series with respect to the depth

$$\varphi = \sum_{n=0}^{\infty} \Phi_n(x,y,t)(H + z)^n. \tag{0.11}$$

By virtue of the condition (0.10) we have $\Phi_1 \equiv 0$ and the recurrence formulae

$$\Phi_{2n+1} = 0, \quad \Phi_{2n+2} = -\frac{\Delta\Phi_{2n}}{(2n+2)(2n+1)} \tag{0.12}$$

then follow from (0.7). Consequently, the only independent function is Φ_0. Substituting (0.11) into the boundary conditions (0.8) and (0.9) and taking (0.12) into account, we obtain the system

$$\frac{\partial\eta}{\partial t} + \nabla\eta\nabla\Phi_0 + (H+\eta)\Delta\Phi_0 - \frac{1}{6}(H+\eta)^3\Delta\Delta\Phi_0 =$$

$$= \frac{1}{2}(H+\eta)^2\nabla\eta\nabla\Delta\Phi_0 + \ldots, \tag{0.13}$$

$$g\eta + \frac{\partial\Phi_0}{\partial t} + \frac{1}{2}(\nabla\Phi_0)^2 - \frac{(H+\eta)^2}{2}\frac{\partial}{\partial t}\Delta\Phi_0 =$$

$$= -\frac{(H+\eta)^2}{2}\nabla\Phi_0\nabla\Delta\Phi_0 + \ldots, \tag{0.14}$$

where the terms containing $(H + \eta)$ in fourth and higher powers are not written out. The system (0.13), (0.14) does not include the vertical coordinate z and derivatives with respect to z. The system is exact (provided all terms are taken into account) indicating that the elimination of the nonwave coordinate may be done in principle for a rather general case. It is clear, however, that the system with an infinite number of terms is not an essential simplification of the initial system. Nevertheless, if the depth H is considerably smaller than the wavelength λ, i.e. $\mu = H^2\lambda^{-2} \ll 1$, and the amplitude η is smaller than the depth, i.e. $\varepsilon = \eta H^{-1} \ll 1$ then assuming $\varepsilon \sim \mu$, all the terms on the right-hand sides are proportional to ε^2, μ^2, $\varepsilon\mu$ or to higher powers of these parameters. Neglecting these terms and introducing the particle velocity $\vec{u} = \nabla\Phi_0 - \frac{1}{2}H^2\nabla\Delta\Phi_0$ we obtain, after a certain transformation, the following system

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\nabla)\vec{u} + g\nabla\eta = 0, \tag{0.15a}$$

$$\frac{\partial\eta}{\partial t} + \operatorname{div}\left\{(H+\eta)\vec{u} + \frac{1}{3}H^3\Delta\vec{u}\right\} = 0. \tag{0.15b}$$