STUDIES IN THE MATHEMATICAL THEORY OF INVENTORY AND PRODUCTION

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STUDIES IN THE MATHEMATICAL THEORY OF INVENTORY AND PRODUCTION

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PREFACE

The research papers in this collection treat a number of mathematical and conceptual problems in the analysis of business decisions about inventories and production. The results presented in these papers have been obtained during the last three years and appear in print for the first time in this volume.

Each chapter is signed by its author or authors and may be read independently of other chapters. We suggest, however, that Chapters 1 and 2 be read first, since they serve to place the following chapters in their proper setting with respect to previous work, and to furnish the reader with a survey of the many common features of all inventory models. In the following chapters, we have frequently omitted detailed explanation of the models whose significance seems to us clearly presented in Chapter 2.

In Chapter 3, the reader will find summaries of the results of the subsequent chapters.

Most of the research reported in this book was done at Stanford University, with the support of the Office of Naval Research. Chapter 12 was prepared at The RAND Corporation. Chapter 13 was prepared at the Cowles Foundation for Research in Economics, Yale University, with the support of the Office of Naval Research. To all these organizations, we wish to express our gratitude.

We also wish to thank Professor Albert H. Bowker, Director of the Applied Mathematics and Statistics Laboratory at Stanford University, both for doing so much to encourage the development of mathematical research in the social sciences at Stanford and for first suggesting that we assemble a book on inventory and production theory. John Gessford and Donald Roberts spent considerable time reading several of the chapters in their first drafts, and made a number of valuable suggestions. Discussions with Ronald Pyke were very helpful in developing some of the work on stochastic processes in this volume. Finally we must thank Caroleanne Roberts, Laura Staggers, and Sharon Steck for converting our appalling handwriting into typescript.

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PART I INTRODUCTION

1

HISTORICAL BACKGROUND

KENNETH J. ARROW

1. Introduction

The last seven years have seen a remarkable development in the study of optimal policies for the holding of inventories. It may be useful to survey briefly the roots of these ideas in earlier thinking in economics and in the business administration literature. As we shall see, most of the individual elements of current inventory models will be found in the earlier literature (up to, say, 1946), though the form was frequently rather imprecise and little was done to integrate them into a consistent framework.

It must be stated immediately that economic theory has had remarkably little to say about inventories. This neglect is partly connected to the emphasis on equilibrium situations, in which the holding of inventories in anticipation of price changes is ruled out by hypothesis. But even under static conditions it is usually agreed that inventories will be held in most circumstances in spite of storage costs and the tying-up of capital which could be invested elsewhere. There must therefore be utilities derived from the holding of inventories which outweigh these costs. Nevertheless, the usual treatise or textbook has only the most scattered references to the motives for the holding of inventories.

There is, however, a very considerable literature on a closely related topic, the demand for money. Firms and individuals ordinarily hold stocks of money, even though they could be invested to earn interest. Though there are many controversies in this area, it is broadly agreed that there are three motives for holding cash, referred to by Keynes ([8], pp. 170–71, 195–96) as the transaction, precautionary, and speculative motives. Roughly speaking, the speculative motive is the possibility of profit through changes in prices and interest rates or other demand and supply conditions, the precautionary motive the need for protection against uncertainty, and the transaction motive the cost of changing rapidly from cash to bonds (or other investments) and back again to meet the needs for cash. Since most (though not all) of the elements in the determination of the level of inventories can be brought under one of these three heads, we shall organize our discussion about them. Reference will be made freely to the literature of monetary theory, as well as to the literature dealing with inventories proper.

Until the more powerful tools of current inventory theory were developed, the choice open to the theorist was between a vague analysis which covered all factors and a precise analysis of an oversimplified model, and it is not surprising that the former was frequently found useful for purposes of overall economic analysis,[1] as opposed to the determination of optimal policies for a single business. However, we shall see examples of the latter also, particularly in regard to the speculative and precautionary motives.

2. The Transaction Motive

Suppose that demand and cost conditions in the future are known with certainty to remain the same as they are today. It is usual in the literature to argue that even under these conditions inventories and cash balances will be held because it is not possible to synchronize perfectly inflow and outflow of goods or money. There are several developments in the analysis, of varying degrees of sophistication and insight.

(a) The simplest is a mechanical view that assumes, without further analysis, that a given volume of sales requires a minimum amount of cash balances or inventories to avoid disruption. This minimum is supposed to be determined by the technical factors which cause the lack of synchronization; the firm cannot maintain the given volume of sales with smaller inventories or cash balances and derives no benefit from having more. Hence, for a given sales volume, the inventories or cash balances held are independent of other variables, in particular of the rate of interest.

It is also assumed in this theory that if the sales of the firm increase, all inflows and outflows increase in the same proportion. Thus it was widely assumed that the transaction demand (the demand attributable to the transaction motive) was proportional to sales and independent of other variables.

(b) This analysis does not really go deeply into the actual alternatives available to the entrepreneur. Let us examine the process more closely in the case of cash balances. There is a sequence of time points at each of which there is a payment or receipt. The time points and the amounts paid or received at each are known in advance. If the firm is not losing money, then over a sufficiently long period the cumulated

[1] In the following discussion, I have relied heavily on the sparkling account of the history of monetary theory given by D. Patinkin ([13], Supplementary Notes C-K).

receipts are at least as much as the cumulated payments. But it may well be that for some time points, the cumulated payments exceed the cumulated receipts (thus a merchant may have to pay for his goods before he has a chance to sell them).

One possible policy for a firm under these circumstances is to lend money (buy bonds) whenever its cumulated receipts exceed its cumulated payments, and borrow money (issue bonds) in the opposite case. Under this policy the firm would maximize its income over time without ever holding cash balances. Consideration of this policy, which is an alternative to simply holding enough cash to meet the maximum possible deficit, shows that mere lack of synchronization is not a sufficient explanation of the holding of cash balances.

It is usual in the literature to argue at this point that buying and selling bonds or other credit instruments in itself involves transfer costs. These costs would outweigh under certain circumstances the gains in interest received by investing whenever possible. Thus in a qualitative way the existence of transfer costs explains the holding of cash balances.

(c) However a precise description of transfer costs in an explicit model is hard to find in the literature. K. Schlesinger [14], writing in 1910, has given perhaps the most careful account of the transaction demand until the last few years; he in effect simply assumes that money cannot be borrowed, at least not within some given time period, so that the cash balance needed is equal to the maximum excess of cumulated payments over cumulated receipts. Such a simple model may be useful as an extreme case, in which transfer costs are infinite at least within a period, but it still leaves the nature of those costs rather indefinite.

(d) For a more specific theory of transfer costs, we must turn to the inventory control literature which developed in the 1920's, apparently partly under the impetus of the very considerable inventory losses suffered by American businessmen in the depression of 1921. The general explanation, as developed by a number of writers, the earliest of whom are Davis [4], Mellen [11], and Owen [12],[2] is the economy of placing larger orders. Specifically, they assumed that in addition to the price paid for the goods ordered, there is a procurement cost to each order which is independent of the magnitude of the order. In that case, there is an incentive not to order continuously but to order larger amounts less often. Such a policy, however, implies the holding of inventory in the intervals between the orderings. To illustrate in the simplest case, let us suppose that there is a procurement cost K for each order, independent of its magnitude. Let s be the sales per annum

[2] For additional references, see Whitin [20], p. 32, fn. 4.

(assumed equally spaced in time), c the cost of purchasing one unit (assumed independent of the amount purchased), h the cost of holding a unit of inventory (including interest on the capital tied up), and x the amount ordered at any one time. Once an amount x is ordered, the inventory is allowed to run down to zero before another order is delivered, so that the average amount of inventory on hand is $x/2$, and cost of holding it is $xh/2$. The number of orders per annum is s/x, so that the procurement cost is Ks/x. Finally, the total purchase cost is cs. Then the problem is to choose x to minimize the sum of these three costs; by differentiation, the optimal amount to be ordered is easily seen to be

$$x^* = \sqrt{\frac{2Ks}{h}}.$$

The importance of the procurement costs is made clear by this formula, since $K = 0$ implies that $x^* = 0$.

More complicated models in which unit storage costs vary and in which there are quantity discounts in purchasing can easily be handled along similar lines; for a summary of such results, see Whitin [20], pp. 33–38.[3]

This model of transfer costs actually leads to conclusions different from those in the vaguer model sketched in (a) above. In particular, (1) the amount of inventory held is, on the average, proportional to the square root of annual sales, rather than proportional to them; (2) the amount of inventory held is not independent of the rate of interest (which enters into h).

(e) More recently Baumol [2] has pointed out the relevance of these inventory results to the transactions demand for cash. However, it may be remarked that in the case of money the intuitive notion of "transfer cost" is probably not too well rendered by a cost independent of the scale of the transaction. It suffices to assume that each purchase or sale of bonds or other credit instruments is accompanied by a brokerage fee proportional to the magnitude of the transaction. Then in a period of temporary cash surplus it may not pay to buy bonds if the period is too short for the interest earned to pay for the brokerage fee. In this case, an increase in the scale of operations will cause a strictly proportional increase in the desired cash balance if a scale increase means simply a uniform increase in each transaction, but the effect is more complicated if the increase takes the form of more frequent transactions (receipts and payments). Also in this case the transactions demand will be affected by the rate of interest, since at higher

[3] Quantity discounts have much the same effect as procurement costs. Mathematically, the critical consideration is that when the cost of ordering is a *concave* function of the amount ordered, i.e., when there are economies of scale, there is a motive for ordering in larger quantities and consequently having inventories.

rates of interest it will pay to buy bonds for shorter intervals of time. An account combining the two possible types of transaction costs has recently been given by Tobin [18].

3. The Precautionary Motive

A good deal of the informal discussion in monetary and inventory theory has stressed the inability to predict cash needs or demands, and the consequent need for a safety allowance against risk. It should be made clear that uncertainty, by itself, would not necessitate the holding of inventories or cash balances, if goods or cash could be obtained instantaneously without extra cost for speedy delivery. Thus it must be assumed either that the considerations of the last section operate or at least that goods and cash can only be obtained after a lag (at least without a premium payment for immediate delivery).[4]

The problem of safety allowances in the face of uncertainty was considered by Edgeworth [5] in 1888 with respect to the determination of reserve ratios for banks. Suppose there are a large number of depositors, for each of whom the amount to be withdrawn during the next period is a random variable, independent of the withdrawals of other depositors. Then the total amount withdrawn is a normally distributed random variable x, with, say, mean μ and standard deviation σ. The bank may regard as tolerable a certain probability p of not being able to meet all withdrawals. Let x_0 be the amount of cash kept on hand. Then $x_0 = \mu + k\sigma$, where k is the normal deviate corresponding to an upper tail of p.

The actual inventory is a random variable, $x_0 - x$. Its expected value is $k\sigma$, and this quantity may be termed the safety allowance. Suppose the number of depositors increases, the new depositors having the same probability distribution for withdrawals as the old ones. Then the standard deviation will increase as the square root of the number of depositors, and the same is true of the safety allowance. We thus get another interesting example of the square root formula, from an entirely different point of view.

Edgeworth took the probability of depletion as a datum. A deeper analysis would try to relate it to more underlying cost factors. In this case, the obvious ones are, on the one hand, some kind of penalty for being unable to meet a requested withdrawal, and, on the other, the possibility of earning interest by lending money instead of keeping it as a reserve. The penalty cost has therefore become an important feature of current models in the inventory field.

Later writers occasionally repeated Edgeworth's arguments but added

[4] Note that lags in delivery were irrelevant in the last section, where certainty is assumed; the time of ordering is simply made early enough to ensure delivery at the needed moment. When the future is certain, this is no problem.