# THE VOLTERRA AND WIENER THEORIES OF NONLINEAR SYSTEMS

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**A Wiley-Interscience Publication**
**JOHN WILEY & SONS**
New York · Chichester · Brisbane · Toronto

**Library of Congress Cataloging in Publication Data:**

Schetzen, Martin.
The Volterra and Wiener theories of nonlinear systems.

Bibliography: p.
Includes index.
1. System analysis. 2. Linear operators.
3. Nonlinear theories. I. Title.

QA402.S3535 003 79-13421
ISBN 0-471-04455-5

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

**To the memory of**
**Norbert Wiener**
**scholar, teacher, and friend**

# Preface

Generally, the modeling used in the study of systems can be classified as being either implicit or explicit. Implicit models are those in which the system response is expressed as an implicit operation on the system input. An example is the modeling of the relation between the system response and input by a differential equation. Explicit models are those in which the system response is expressed as an explicit operation on the system input. An example is the modeling of the relation between the system response and input by a convolution integral. Neither type of model is all encompassing and each provides insights not provided by the other. The model that is best to use thus depends on the specific questions being asked and the specific understanding of the system operation being sought. For example, the study of system oscillations normally is best achieved using implicit models whereas the study of the spectrum of the system response for a random input normally is best achieved using explicit models. The Volterra and Wiener theories of nonlinear systems are explicit models.

One major area of study for which these theories are ideally suited is the modeling of "black boxes." Many problems in analysis, design, and control require accurate models of the systems involved. Sometimes sufficient information concerning a system is available so that a set of equations that accurately model the system can be derived. More often, a system is available only as a "black box" so that the relation between the system input and output is not so derivable. This is especially the case for many nonlinear physical and biological systems. The Volterra and Wiener theories often can be used to analyze such systems. In this book, the modeling of "black boxes" by these theories is discussed in detail. Also the various insights and simplifications in nonlinear system analysis, characterization, and synthesis provided by these theories is presented as part of their development.

Since this is the first book to be written on this subject, the organization, development, and the interpretation of the material presented is, to a large extent, the outgrowth of my own research. However, the influence of my teachers Y. W. Lee and N. Wiener is present throughout the text. The Volterra theory is developed in the first six chapters of the text. It is developed in a new way from the viewpoint of $p$-linear operators. This simplifies and lends a great deal of insight to the theory. Some applications of the Volterra theory to the study of nonlinear differential equations, nonlinear feedback, and the inverse of a nonlinear system is then discussed in Chapters 7 and 8. Only the basic concepts and methods are discussed together with some numerical illustrations. Details of specific applications of the Volterra series can be obtained from the references listed in Appendix B. The Wiener theory of nonlinear systems is then motivated and developed in Chapters 9–14. In Chapter 15, the Wiener theory is used as the basis of a method for determining optimum nonlinear models of a given or desired system. To develop a deeper insight into the Wiener theory, a detailed decomposition of the Wiener characterization of a nonlinear system, called the Wiener model, is developed and analyzed in Chapters 16–20. Many of the advantages of the Wiener model derive from its orthogonal properties when the input is a Gaussian waveform. Using the insights gained from the Wiener model, I develop in Chapter 21 another type of model based on the gate functions, which are orthogonal irrespective of the input waveform. This generalization of the gate model over the Wiener model is, however, obtained at a cost. The cost is that a larger number of coefficients must be determined in order to specify the model. In the final chapter, this cost is analyzed and methods of minimizing this cost for either the gate or the Wiener model are developed.

The Volterra and Wiener theories have been shown to be useful in many areas of study. A number of these areas are referenced in Appendix B. In consequence this text has been written in a manner that is comprehensible to a broad scientific audience for individual study or classroom use at the first year graduate level. An undergraduate course on linear system theory is all that is required to begin the study of this text. An elementary background on random process theory is useful in the study of the second part of this text on the Wiener theory of nonlinear systems. If the reader does not possess this background, it can be obtained from one of the many available books on random process theory or on communication theory.

In order to facilitate the study of this text, a general concept often is introduced by first presenting examples of various orders of complexity to help the reader better visualize the general result being developed. Also, in developing certain methods of analysis, the "brute force" approach is presented first. From this, some insights are obtained that are then used to develop a simpler method from which a better comprehension of the underlying analytical structure is obtained. The problems I composed for this text have been designed to aid the reader in developing a better understanding of the material presented. Thus some problems are simply exercises in utilizing the concepts developed or to fill in some details omitted in the text. Other problems, however, further develop some of the concepts and theories developed in the text. These problems are important since some new useful results are developed in a number of them. Working through these problems with the guidance given will aid the reader to obtain a deeper comprehension of the material presented.

No references are appended at the end of each chapter. Rather, to give a historical perspective of the development of the Volterra and Wiener theories and to properly credit the major contributors to the theories, I have written a historical bibliography which is included in Appendix B. References to further developments of the material discussed in this text is included in the bibliography. However, there is much new material included which I developed in the process of writing this text and have not previously published.

It is a pleasure for me to express my thanks to Mr. Janusz Sciegienny who did all the programming for the computer plots in Chapter 8 and to Dr. Cynthia Whitney who read a draft of the manuscript and made a number of valuable suggestions. I am especially grateful to Leslie M. Herman who cheerfully typed a draft of the manuscript from my handwritten copy and to Doris R. Simpson for her competent and patient typing of the entire manuscript. Finally, my heartfelt thanks to Jeannine Desrochers, who helped me with the tedious task of proofreading the galley and page proofs. This book was completed while I was a visiting professor in the Electrical Engineering and Computer Sciences Department of the University of California, Berkeley. The friendly and intellectually stimulating atmosphere there contributed greatly to my completing this text on which I had been working for some years.

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*Boston, Massachusetts*
*September 1979*

# Contents

# ONE
# General Introduction

## 1.1 BASIC SYSTEM CONSIDERATIONS

A system can be defined in a mathematical sense as a rule by which an excitation $x$ is mapped into a response $y$. The rule can be expressed symbolically as

$$y = \mathbf{T}[x] \tag{1.1-1}$$

In scientific applications, the response and the excitation usually are functions of an independent variable, such as position or time. If they are only functions of time, $t$, the expression in eq. 1.1-1 is written in the form

$$y(t) = \mathbf{T}[x(t)] \tag{1.1-2}$$

In this representation, **T** is called an operator because the response function $y(t)$ can be viewed as being produced by an operation on the input function $x(t)$. The statement that $y(t)$ is the response of a system to the excitation $x(t)$ means that there exists an operator **T** or, equivalently, a rule by which any given input $x(t)$ is mapped into a unique output $y(t)$.

The mapping can be viewed as shown in Figure 1.1-1 where the set of all possible inputs is denoted by $X$ and the set of all possible outputs is denoted by $Y$. In the figure, the time functions denoted by $x_1$ and $x_3$ are both mapped into the same output $y_1$. Note however, that one input can not be mapped into more than one output. A system operator thus constitutes a many-to-one mapping. For example, a square-law device characterized by the mapping $y(t) = x^2(t)$ is a system; however, the time

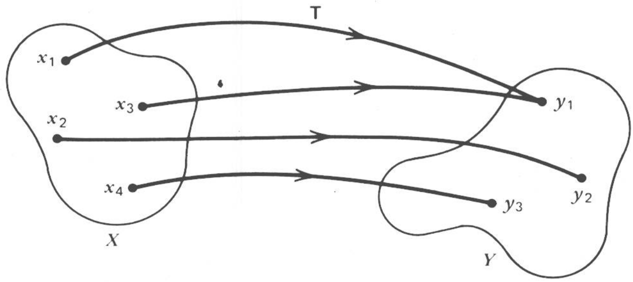


**Figure 1.1-1** The mapping of operator **T**.

function $x(t)$ cannot be recovered from $y(t)$ by a system, since it would require the mapping of one input into more than one output. The reason for the impossibility of such a system is that, in taking the square root of $y(t)$, there is no general rule by which the correct sign of $x(t)$ can be known from only $y(t)$. Thus there is no system that is the inverse of a square-law device. In general, we note that a system that is the invers^ of a given system will exist if and only if the system operator **T** is ; one-to-one mapping.

Two major aspects of system theory are analysis and synthesis. In analysis, the operator **T** of a given or desired system is determined. In synthesis, a system with a given operator is constructed from a given set of elemental systems. If nothing is known a priori concerning the system operator **T**, then analysis can only consist of the construction of a list of input functions and the corresponding output functions. Nothing, however, could be said concerning the output function corresponding to an input function that is not on the list. Further, with only such a list available, synthesis would be, to say the least, extremely difficult. Thus, even if the construction of such a list were possible, it would not be very useful. In order to do meaningful analysis and synthesis then, we require some a priori knowledge of the system.

To fulfill the above requirements, system operators are categorized into various classes. All operators that are members of a given class have certain common properties that form the bases for analysis and synthesis procedures. The required a priori knowledge is whether a system can be considered to be a member of a particular class. The particular class is not unique; rather, the choice of a class is determined by the objectives of the analysis or synthesis. One class that encompasses a very large number of physical systems of importance is that described by Wiener. Roughly this class includes all time-invariant physical systems with noninfinite memory. An example of a system with infinite memory—and thus not included in the Wiener class—is a fuse that will never forget that the current through it exceeded its rating. In this book, we develop the theory of the Wiener class, which is the Wiener theory of nonlinear systems. This theory is based on an orthogonalization of a specific, complete set of time-invariant operators, $\mathbf{H}_n$, called the Volterra operators. The comprehension of the Wiener theory, its implications, and its applications requires a basic understanding of these Volterra operators. Thus we begin with a development of the Volterra theory.

## 1.2 TIME-INVARIANT AND TIME-VARYING SYSTEMS

In the last section, Volterra operators were referred to as time-invariant operators. We discuss exactly what is meant by a time-invariant operator in this section. Consider the system shown in Figure 1.2-1 with the output $y(t)$ for the input $x(t)$. A time-invariant system is one for which the operator $\mathbf{T}$ does not vary with time so that a time translation of the input results in the same time translation of the output. In mathematical terms, we let

$$y_1(t) = \mathbf{T}[x_1(t)] \tag{1.2-1}$$

The system is time invariant if for the input

$$x_2(t) = x_1(t + \tau) \tag{1.2-2}$$

the output is

$$\begin{aligned} y_2(t) &= \mathbf{T}[x_2(t)] \\ &= \mathbf{T}[x_1(t + \tau)] \\ &= y_1(t + \tau) \end{aligned} \tag{1.2-3}$$

For example, consider a time-invariant system with the output $y_1(t)$ for the input $x_1(t)$ shown in Figure 1.2-2. Then, for the input $x_1(t + \tau)$, the output is $y_1(t + \tau)$ as shown in the figure. For a system to be time invariant, eq. 1.2-3 must be true for *any* function $x_1(t)$ that can be considered as an input to the system. A system that is not time invariant is said to be a time-varying system.

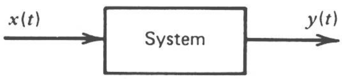


**Figure 1.2-1** Schematic representation of a system.

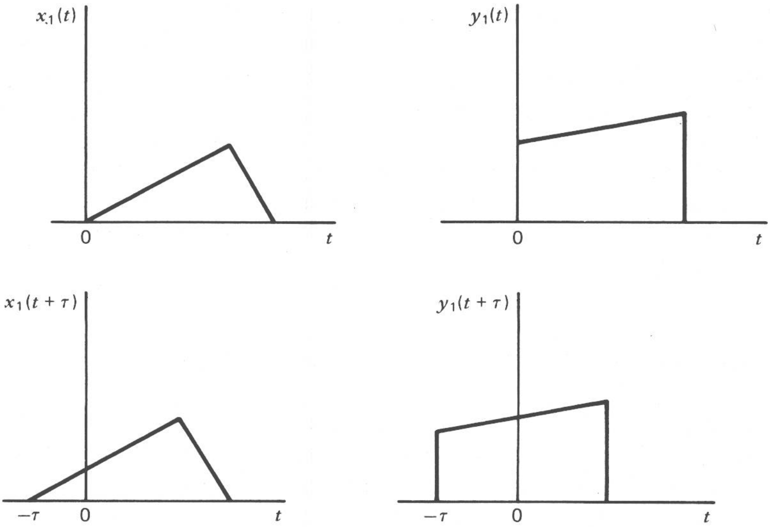


**Figure 1.2-2** The input–output relationship of a time-invariant system.

As an illustration of the application of eq. 1.2-3 to a particular system, consider the resistor circuit shown in Figure 1.2-3. For the system shown, the input is the voltage $v_i(t)$ and the corresponding output is the voltage $v_0(t)$. The relation between the output and the input is

$$v_0(t) = \frac{R_2(t)}{R_1(t) + R_2(t)} v_i(t) \tag{1.2-4}$$

The output for the input $v_2(t) = v_i(t + \tau)$ is

$$\begin{aligned} y_2(t) &= \frac{R_2(t)}{R_1(t) + R_2(t)} v_2(t) \\ &= \frac{R_2(t)}{R_1(t) + R_2(t)} v_i(t + \tau) \end{aligned} \tag{1.2-5}$$

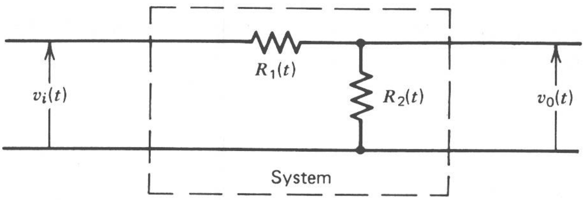


**Figure 1.2-3** Circuit for a time-varying system.