

OPERATOR ALGEBRAS AND QUANTUM FIELD THEORY

EDITORS

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 International Press

0413.3-53
061
1996

9930208



Operator Algebras and Quantum Field Theory

Accademia Nazionale dei Lincei
Roma, July 1-6, 1996

香港方樹福堂
基金贈書



E9960208

Editors:

S. Doplicher, R. Longo, J.E. Roberts, L. Zsido

ACKNOWLEDGEMENTS

The Conference has been hosted by the Accademia Nazionale dei Lincei and, on the closing day, by the Mathematics Department of the University of Rome “La Sapienza”. It has been rendered possible by the help and financial contribution of several Institutions, in particular:

The European Union through its HCM programme

The Italian National Research Council CNR, both through its “Comitato per la Matematica” and through the “Gruppo Nazionale per l’Analisi Funzionale ed Applicazioni”

The University of Rome “La Sapienza”

The Centro Linceo Interdisciplinare

The University of Rome “Tor Vergata”

The contribution of EU was channelled through the network Operator Algebras coordinated by Gert Pedersen; the existence of this network and the possibility it offered of organizing meetings, the first held in Cork in 1995, provided the initial impulse for organizing this conference. We would like to record our deep gratitude to our Colleague Gert Pedersen for his unselfish dedication to the highly nontrivial organisational problems involved in the network; his exceptional competence and passionate engagement were the backbone, both of this, and of the succeeding network on Non-Commutative Geometry. May “the spirit of Copenhagen” long continue to inspire us.

Our Masters thesis and PhD students, postdocs and young researchers here in Rome have been a great help both before and during the conference; we would like to thank them all. But it is a pleasure to express our special thanks to Daniele Guido and Tommaso Isola for their continuing help, in particular in respect of the divers problems related to the task of assembling the Proceedings.

Two of our secretaries, Roberta Diecetti and Carla Pietrini, coped valiantly with the task of arranging accommodation for participants and dealt with the practical problems during the conference itself. A third, Simonetta De Nicola, has eased our burden with \TeX whilst editing these proceedings. We thank each of them for their timely help in these critical phases of a conference.

PROGRAMME

Monday July 1, 1996. Morning Session. Chairman: R. Kadison

9.30 - 10.00 Opening address

10.15 - 11.15 D. Voiculescu: *Topics in Free Entropy*

11.45 - 12.45 F. Radulescu: *Berezin Quantization and Type II₁ Factors*

Afternoon Session. Chairman: E. Wichmann

15.00 - 16.00 R. Verch: *Scaling Algebras, the Renormalization Group and the Principle of Local Stability in AQFT*

16.15 - 17.15 K. Fredenhagen: *Interacting Quantum Fields on Curved Space-time*

17.45 - 18.45 A. Ocneanu: *Modular Invariants, Subfactors and Quantum Field Theory*

Tuesday July 2, 1996. Morning Session. Chairman: D. Kastler

9.00 - 10.00 D. Bisch: *Algebras of Symmetries Associated to Subfactors*

10.15 - 11.15 M. Izumi: *Galois Correspondence for Compact Group Actions on Factors*

11.45 - 12.15 J. Bros: *The pole-particle concept in Quantum Field Theory*

12.15 - 12.45 C. D'Antoni: *Chiral nets with diffeomorphism covariance*

Afternoon Session. Chairman: R. Douglas

15.00 - 16.00 M. Landstad: *Some Quantum Groups and Manifolds*

16.15 - 16.45 H. Kosaki: *The Group of One-Dimensional Bimodules Arising from Composition of Subfactors*

16.45 - 17.15 M. Pimsner: *Embedding covariance algebras of flows into AF-algebras*

17.45 - 18.45 M. Takesaki: *The Cocycle Conjugacy of Amenable Group Actions on AFD Factors of Type III₁*

Wednesday July 3, 1996. Morning Session. Chairman: N. Hugenholtz

9.00 - 10.00 H.J. Borchers: *Half-Sided Modular Inclusions and Structure Analysis in Quantum Field Theory*

10.15 - 11.15 D. E. Evans: *The Rohlin Property and Outer Conjugacy Classes of Trace Scaling Automorphisms of Stable AF Algebras*

11.45 - 12.45 A. Connes: *Gravity Coupled with Matter and the Spectral Action Principle*

Thursday July 4, 1996. Morning Session. Chairman: R. Powers

9.00 - 10.00 O. Bratteli: *Translationally Invariant States on Quantum Lattice Systems and Representations of Cuntz Algebras*

10.15 - 11.15 M. Rørdam: *Density of the Invertible Elements in Reduced Free Products*

11.45 - 12.45 U. Haagerup: *Orthogonal MASA's in the $n \times n$ Matrices, Complex Hadamard Matrices and Cyclic n -Roots*

Afternoon Session. Chairman: B. Schroer

15.00 - 16.00 H.W. Wiesbrock: *Modular Inclusions and Intersections of Algebras in Quantum Field Theory*

16.15 - 17.15 K.H. Rehren: *The Generalized Cuntz Algebra of Charged Fields*

17.45 - 18.15 K. Szlachányi: *Weak C^* -Hopf Algebras*

18.15 - 18.45 S. Summers: *Bell's Inequalities and Algebraic Invariants*

Friday July 5, 1996. Morning Session. Chairman: R. Nest

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10.15 - 11.15 E. Størmer: *On Noncommutative Entropy*

11.45 - 12.45 S. Popa: *On the Notions of Amenability for Subfactors and for their Invariants*

Afternoon Session. Chairman: C. Anantharaman-Delaroche

14.20 - 15.00 D. Kastler: *On the Connes-Chamseddine universal action*

15.00 - 15.30 D. Guido: *Asymptotic Dimension and Novikov-Shubin Invariants for Open Manifolds*

15.30 - 16.00 C. Pinzari: *Simplicity of the Cuntz-Krieger-Pimsner C^* -algebras*

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17.45 - 18.45 G. Elliott: *On the Construction of Simple C^* -Algebras with Specified Invariants*

Saturday July 6, 1996 Morning Session. Chairman: A. Helemskii

9.00 - 10.00 Y. Kawahigashi: *Quantum Doubles, Orbifold Subfactors and Braiding*

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11.45 - 12.45 G. Skandalis: *Multiplicative Unitaries*

Afternoon Session. Chairman: J. Renault

15.00 - 16.00 D. P. Blecher: *An Operator Space Approach to Operator Algebras and Hilbert Modules*

16.15 - 17.15 A. Wassermann: *Algebraic Conformal Field Theory*

17.45 - 18.45 G. Pedersen: *Extensions of C^* -Algebras*

ORGANIZING COMMITTEE

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POUR DANIEL

This Conference was dedicated to our dearest friend Daniel Kastler in celebration of his Seventieth Birthday. We all know how central Daniel has been to developments in this subject. These Proceedings are a token of our deep gratitude and friendship.

At the Conference dinner a short ceremony took place to honour Daniel Kastler; some of his friends said a few words, more precisely, introduced by Sergio Doplicher:

Alain Connes; Klaus Fredenhagen read a letter of Rudolf Haag; Nico Hugenholtz; Dick Kadison; Masamichi Takesaki; Erling Størmer; Gert Pedersen; John Roberts; Joachim Cuntz; Adrian Ocneanu; Daniel Testard.

It was not planned to have written texts, but we are pleased to reproduce those we chance to have (Doplicher, Connes, Haag and Takesaki) to mark the memory of that occasion and of the many nice words spoken for Daniel.

Sergio Doplicher Roberto Longo John Roberts László Zsidó

Memories from our youth are often like sunset clouds, even if dark they are sweetened by a golden skyline. From a nice period of my life like that of my early visits to Bandol, memories are real golden jewels: the "Bandol Institute" had no building, so Daniel would sometimes organize talks in the Elementary School, where we shared those tiny benches with famous scholars talking von Neumann Algebras under the eye of children's drawings.

Memories of our first collaboration on Covariance Algebras are intertwined with the colours of Provence and with the dark blue of its sea: we would work in Daniel's living room, writing with chalk on the window panes so that we stared at our C^ -algebras as they floated over a landscape spanned by the Mistral.*

Lisl would not complain but would rather find our dusty traces decorative, and would keep us there at dinner after each session.

But another Mistral blows through those memories of more than three decades ago: Daniel's inextinguishable enthusiasm, whose frequent bursts kept up everybody else's spirits; between those bursts he would move to the piano to try out the opening of Beethoven's third concerto, interrupting the nap of one of his many cats sleeping on the scores.

A few unforgettable years in Bandol followed intermittently by several short stays, spotting my life as little dreams, have seen my aging and many things changing, except one: Daniel's incredible enthusiasm, ever fresher and more lively.

Let this charming fresh Mistral long blow through our community! Heartfelt thanks, dear Daniel, for all you have done and will continue to do for Science and for our Community!

Heartfelt thanks, dear Daniel and Lisl, for the invaluable treasure of your friendship!

Sergio

Cher Daniel,

Permetts moi, en cette hereuse occasion de te saluer, de saluer ta noblesse d'esprit, ton charme italien qui m'a seduit dès notre première rencontre à Seattle, il y a 25 ans et qui est le premier aspect de ta personnalité que l'on perçoit avant de rencontrer cette rigueur toute germanique qui soustend tous tes travaux scientifiques.

Laisse moi, tant que tu n'a pas droit à la parole (ce dont tu dois souffrir), saluer ton approche metaphysique des problèmes et ton enthousiasme pour la connaissance scientifique, notre seule arme contre le passage du temps.

Il m'est arrivé plusieurs fois de lire tant de signes négatifs dans les augures que le découragement n'était pas loin. À chaque fois j'ai trouvé refuge et reconfort dans ton attitude, désintéressée, passionnée et pure.

En un sens faire de la recherche c'est revenir autant que possible vers cet etat d'innocence où l'on est, enfant, animé de cette foi profonde dans le sens de l'existence. Tu m'a permis de comprendre, loin des froideurs parisiennes, à quel point l'art est inseparables de la science, la profonde vérité de ces résonances subtiles entre l'art comme ouverture de nature poétique sur l'imaginaire et la science, si éloignée de la sécheresse d'adorateurs du point de vue materialiste.

J'etais recemment à un congrès à Oxford où, parlant de Roger Penrose et des differentes approches aux problèmes de la physique, Atiyah distinguait clairement le clan des sceptiques de celui des mystiques. J'ai appris avec bonheur qu'il se rangeait, comme toi, dans celui des mystiques.

Il m'est arrivé de percevoir dans ta voix une infinie tristesse de ne pouvoir rien contre l'immense reservoir de scepticisme de suffisance et d'incompréhension que renferme l'espèce humaine. J'éprouve comme une chance rare le plaisir de te connaître et de naviguer à tes cotés dans notre periple scientifique commun.

Alain

Dear Daniel,

When I learned of the plan of the conference organizers to celebrate your 70th birthday by a splendid dinner I was tempted to come to Rome to participate. Sober reflections on important business, such as guaranteeing an adequate amount of sleep, finally let me decide against it.

There will be many friends gathered on this occasion, many of them being also my friends. Greetings to them all.

Thinking of you, Daniel, several remarkable traits come immediately to mind. There is DANIEL THE FIGHTER. The memory goes back to 1955 when, trying to procure a professorship in thermodynamics for me at Saarbrücken, you organized an uprising of the young Turks against the establishment for this purpose. I remember your extended battles with the Dean in Marseille St. Charles in 1959, trying to obtain appointments for the best students of Laurent Schwartz. I remember countless subsequent battles for and within Luminy, the latter, I am afraid still going strong.

There is DANIEL THE MANY SIDED ARTIST with his love for music and mathematics, his lively interest in painting, architecture, in fact in almost all cultural developments.

Inseparable from the fighter and the artist there is DANIEL THE ENTHUSIAST. He, the enthusiast, provided the resources and the toughness for exploits like the

assimilation of vast areas of modern mathematics at an age beyond 50. He also invoked holy anger against those who do not recognize beauty when they see it or who are even too stupid to see where the wind is blowing from while they watch Don Quichote fighting the windmill. (This is an adaptation from the unwritten tid-bits of Daniel Kastler).

Paramount, however, is DANIEL THE FRIEND. Kindness, hospitality, generosity, help when help is needed can be relied upon as a matter of course. But here we have to include Lisl. In German the wife is often referred to as the better half. So let me propose a toast to both halves of Daniel.

All the best wishes for the next decade!

Rudolf

To Professor Daniel Kastler and Mrs. Lisl Kastler

Dear Daniel and Lisl, Ladies and Gentleman:

I am very honoured to be here speaking to you, in particular to Daniel. Daniel, Lisl and their family have been close friends with Kyoko, myself and Yuki for over twenty years. It is a long period of time, but it does not seem that way. I feel it was just yesterday when I spent my first year in France in Bandol from 1973 to 1974.

When I met Daniel in Baton Rouge in the Spring of 67, I recognized immediately his extraordinary charming personality, as a scientist, as a teacher, as an artist and as a human being. Of course, I had known his name through reading his articles on the famous Haag - Kastler C^ -algebraic approach to quantum physics, although my knowledge of physics was, and still is, scant.*

It is impossible for me to describe how much Kyoko and myself learned from Daniel and Lisl, and how much we admire them. Daniel's energy and enthusiasm for new ideas in theoretical physics and relevant mathematics is incredible. He never takes a safe approach to anything. He loves challenges. I can only imagine how dangerously adventurous he must have been when he was young. What I know is a rather limited number of charming stories which Lisl has told us. As a skier, I cannot forget the episode on how he skied with beautiful Tyrolean girls on the terrifying Austrian slopes. It was his first skiing experience. Fearless in the face of challenge, he skied straight and naturally fell and broke his pole. He refused to quit and borrowed a new pole from an accompanying Tyrolean girl. Events repeated themselves. But he kept on skiing and finally broke all of the ski poles of Lisl's friends on the Austrian Alps.

Probably, what characterizes Daniel is his hunger for truth in every aspect of human life. Also Lisl's dedication to Daniel's mission and her warm hospitality extending to all who worked with Daniel are an integral part of our happy memory of Bandol. For a teacher and researcher, curiosity and the search for truth are the most important requirements. But, Daniel goes much further than that. His journey to seek the truth goes really deep and is decorated with colorful charming episodes.

He has shown incredible creativity in many areas in research of course but he also recognizes the hidden potential in other areas and encourages their pursuit. His generosity and the infectious infusion of his own ideas into others is unparalleled. I don't know how many of us have received his hidden and often

unrecognized help and encouragement. It was Daniel who made the joint work of Connes - Takesaki on the Flow of Weights possible bringing Alain, myself AND two of my former research students, Hiro Takai and Trond Digernes, together to Marseille - Bandol.

As everybody knows, Daniel and Lisl together established the Marseille School of Operator Algebraic Physics located at Bandol. Some people called it the BANDOL FREE UNIVERSITY. Almost all of the senior people in our field spent some time in Marseille - Bandol. Countless results have been obtained there. It would be an interesting project for a mathematical historian to dig out all such results and present his findings to the Mayor of Bandol. As far as my involvement goes, I can mention: Winnink - Takesaki on local normality, Connes - Takesaki on the flow of weights, Olesen - Pedersen - Takesaki on compact abelian ergodic group actions, Nakagami - Takesaki on duality for crossed products and finally Kastler - Takesaki on the KMS condition and duality.

I hope that Daniel's retirement, while completing one distinguished chapter of the Bandol Free University, does not mean the end of the Glory of the School.

The face of operator algebras and of mathematical physics have been changing rapidly. The torch you, Daniel, have lit has been and will be carried to the next brightness by the younger generation.

You and Lisl should be proud of your glorious achievements. I am very honoured to be your friend. Please take care of yourself and enjoy healthy and creative life without interruption. I hope you and Lisl will continue to come to gatherings of this nature in the future.

With love,

Masamichi

FOREWORD

Operator Algebras originated in the work of von Neumann, in particular in his search for a natural mathematical frame for Quantum Mechanics, and in the work of Gelfand and Naimark. They can now be seen as a discipline encompassing Noncommutative Analysis, Geometry and Topology. After the seminal work and ideas of Haag in the Fifties and of Araki, Haag and Kastler in the early Sixties, the original link with quantum mechanics has evolved into a much deeper relation with Quantum Field Theory, thus furnishing a unique example for the mutual interaction and motivation between pure Mathematics on the one hand and rigorous Mathematical Physics on the other.

Nowadays Operator Algebras is undoubtedly one of the domains in Mathematics most notable for the depth of its problems, the richness of its new ideas, its connections to so many different fields and its potentiality as a unifying language and source of illumination.

For these reasons, a Conference covering all aspects of the field would have much exceeded the limitations imposed by the facilities and finances at our disposal. Yet we tried to provide a broad coverage of the field seeing that the last such conference in Rome was convened 21 years ago.

Nonetheless, several major developments were not represented; to mention just a few, the relations with number theory described by the remarkable non-commutative dynamical system with phase transitions intrinsically associated with prime numbers by Connes and Bost; the developments on the Novikov conjecture and the Baum-Connes conjecture; the far-reaching generalizations of the Atiyah-Singer index theorem; the unravelling of Banach space aspects.

These limitations have been mitigated by including as part of the Proceedings a few invited contributions not corresponding to actual talks. On the other hand, some of the speakers were not able to submit a text for the Proceedings. For the record and to bear witness to current activity in the subject, we have reproduced here the list of talks actually delivered.

As a help towards orienting the reader, we have divided the book into chapters. Naturally, this has led us, in a number of cases, to make what are rather arbitrary divisions.

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I

C^* -Algebras and their Invariants

EXTENSIONS OF C^* -ALGEBRAS

GERT K. PEDERSEN

Institute of Mathematics
University of Copenhagen

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INTRODUCTION

The theory of extensions of C^* -algebras has classical roots, preceding the formal theory of C^* -algebras by several decades. If \mathfrak{H} denotes the separable, infinite-dimensional Hilbert space, and \mathbb{K} the closed ideal of compact operators in $\mathbb{B}(\mathfrak{H})$, then we have a short exact sequence of $*$ -homomorphisms (i.e. an extension)

$$0 \longrightarrow \mathbb{K} \xrightarrow{\iota} \mathbb{B}(\mathfrak{H}) \xrightarrow{\pi} \mathbb{Q} \longrightarrow 0,$$

where \mathbb{Q} denotes the Calkin algebra $\mathbb{B}(\mathfrak{H})/\mathbb{K}$. The earliest problems in operator theory center around the fact that some elements x in $\mathbb{B}(\mathfrak{H})$, notably the unilateral shift operator, are neither invertible nor normal, even though their images $\pi(x)$ in \mathbb{Q} have these properties. And there are no compact perturbations $x + c$, $c \in \mathbb{K}$, that remedy these defects. Phrased differently, there are invertible and/or normal elements in \mathbb{Q} that can not be lifted to invertible and/or normal

elements in $\mathbb{B}(\mathfrak{H})$. The class of *Fredholm operators*, $\mathbb{F}(\mathfrak{H})$, is defined precisely as $\pi^{-1}(\mathbb{Q}^{-1})$, and by Atkinson's theorem $x \in \mathbb{F}(\mathfrak{H})$ iff x (hence also x^*) has closed, cofinite-dimensional range and finite-dimensional kernel. The Fredholm index

$$\text{index}(x) = \dim \ker x - \dim \ker x^*$$

is therefore an integer, and this index turns out to be the only obstruction to invertibility, i.e. to the existence of a compact perturbation $x + c$ in $\mathbb{B}(\mathfrak{H})^{-1}$.

The *Toeplitz algebra* \mathcal{T} generated by the unilateral shift operator is the pre-eminent example of a non-trivial, separable extension

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0.$$

Via the representation $f \rightarrow T_f$ of $C(S^1)$ as *Toeplitz operators* in \mathcal{T} , i.e. as compressed multiplication operators (when \mathfrak{H} is identified with $\ell^2(\mathbb{N})$ and $L^2(S^1)$ with $\ell^2(\mathbb{Z})$ in the standard fashion), this extension has profound links to classical problems in harmonic analysis and complex function theory.

The abstract theory of extensions began in 1967 with Busby's thesis [16]. The present author became involved in 1973 through work with Akemann and Tomiyama on multipliers of C^* -algebras, [3]. Subsequent papers with these sterling friends and others, notably Olsen, see e.g. [2], [40], centered around the study of lifting problems from quotient algebras in general and from corona C^* -algebras in particular. An expository account can be found in [43].

Associating an index map to extensions other than those of the compact operators was first accomplished by Breuer, [8]. His work was generalized by Olsen to cover any von Neumann algebra extension of one of its ideals, [39]. For C^* -algebras the index theory quickly became part of K -theory; the boundary map $\partial : K_1(A/I) \rightarrow K_0(I)$ from the six term exact sequence associated with an extension $I \rightarrow A \rightarrow A/I$ taking the rôle of index for "Fredholm elements" in A . That the other boundary map $\partial : K_0(A/I) \rightarrow K_1(I)$ could be used to encode the obstruction for lifting projections from A/I to A was demonstrated by Brown in the AF case, [9], and later extended to the more general situation involving C^* -algebras of real rank zero, [13]. The recent theory of extremally rich C^* -algebras, cf. [14], seems to provide the perfect setting of index theory for "Fredholm elements" (and even "quasi-Fredholm elements") in a general C^* -algebra relative to an extremally rich ideal, see [15].

The credit goes to Loring for pointing out that lifting problems are better handled by considering the universal C^* -algebra U for the particular relation under consideration, and then looking at the problem of lifting $*$ -homomorphisms, i.e. completing diagrams of the form

$$\begin{array}{ccc} & & A \\ & \nearrow \varphi & \downarrow \pi \\ U & \xrightarrow{\varphi} & B \end{array}$$

where π is a quotient map. Loring's version of the story is available in [34] and comes highly recommended. Our partnership, later strengthened with Eilers, has resulted in a series of papers, [35], [36], [37], [19], [20], [21], [22], and I

am deeply grateful to my young and vigorous coauthors for having prolonged my research activities beyond normal old age. These notes were written in an attempt to give an overview of the part of our work directly connected with extensions; a chart for navigating in ELP-territory. Space and time have not permitted to cover also the applications to non-commutative CW -complexes, although these were one of the main motivations behind our efforts.

This article is an expanded version of the notes for a lecture given at the international conference on "Operator Algebras and Quantum Field Theory" in Rome, July 1–6, 1996. The author wishes to thank the organizers, Sergio Doplicher, Roberto Longo, John Roberts and Laszlo Zsido, for the opportunity to address the spirited audience of present and future experts, and for providing the splendid facilities in the heart of the Eternal City.

2. ABSTRACT NONSENSE

2.1. Categories. We shall find it necessary to operate with three different categories of C^* -algebras, \mathcal{C}_1^* , \mathcal{C}_0^* and \mathcal{C}_p^* .

\mathcal{C}_1^* is the category of unital C^* -algebras with unit-preserving $*$ -homomorphisms as morphisms.

\mathcal{C}_0^* is the category of arbitrary C^* -algebras with arbitrary $*$ -homomorphisms as morphisms.

\mathcal{C}_p^* is the category of arbitrary C^* -algebras with proper $*$ -homomorphisms as morphisms.

Recall from [19], 2.1 that a $*$ -homomorphism $\rho : A \rightarrow B$ is *proper* if any of the following equivalent conditions are satisfied:

- (i) $\rho(A)$ contains an approximate unit for B ;
- (ii) If (u_λ) is an approximate unit for A , then $(\rho(u_\lambda))$ is an approximate unit for B ;
- (iii) $\rho(A)$ is not contained in any proper hereditary C^* -subalgebra of B ;
- (iv) $B = \rho(A)B\rho(A)$ — quite literally.

Whereas the first two categories are well known, the third is a more recent invention. To explain its use we must look at the commutative case. Thus we consider the three subcategories $\mathcal{C}_1^*(\text{abelian})$, $\mathcal{C}_0^*(\text{abelian})$ and $\mathcal{C}_p^*(\text{abelian})$, where only commutative C^* -algebras are allowed, and apply Gelfand theory to find the dual categories of topological spaces. Evidently $\mathcal{C}_1^*(\text{abelian})$ is dual to the category of compact Hausdorff spaces with continuous maps as morphisms. A moment's reflection shows that $\mathcal{C}_p^*(\text{abelian})$ is dual to the category of locally compact Hausdorff spaces with proper continuous maps as morphisms (maps $f : X \rightarrow Y$ such that $f^{-1}(K)$ is compact in X for every compact subset K of Y).

The surprise is, rather, that $\mathcal{C}_0^*(\text{abelian})$ has no obvious dual category. One may be devised, though, by considering as objects the set of locally compact Hausdorff spaces X , each with a point ∞ adjoined, and topologized such that $X \cup \{\infty\}$ is compact. (Thus ∞ is an isolated point iff X is already compact.) The morphisms in this category are the continuous maps $f : X \cup \{\infty\} \rightarrow Y \cup \{\infty\}$, such that $f(\infty) = \infty$. (Thus $f^{-1}(\infty) \setminus \{\infty\}$ represents the part of X that