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# Dynamical Systems and Probabilistic Methods in Partial Differential Equations

1994 Summer Seminar
on Dynamical Systems and
Probabilistic Methods for Nonlinear Waves
June 20–July 1, 1994
MSRI, Berkeley, CA

Percy Deift
C. David Levermore
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# Dynamical Systems and Probabilistic Methods in Partial Differential Equations

# Preface

This volume contains some of the lectures presented at the 1994 AMS/SIAM Summer Seminar, held June 20–July 1 at the Mathematical Sciences Research Institute in Berkeley. It was the intent of the organizers of the summer seminar to introduce the participants to as many of the interesting and active applications of dynamical systems to problems in applied mathematics as the time constraints of the workshop allowed. Consequently, this book covers a great deal of ground. Nonetheless, the pedagogical orientation of the lectures has been retained in this volume, and as such, we hope that it will serve as an ideal introduction to these varied and interesting topics.

While the focus of the workshop was quite broad, several organizing principles emerged. The first was the increasing role of dynamical systems theory in our understanding of partial differential equations. The first three contributions of the present volume are devoted to this theme. In particular, all of these lecturers stressed the importance that the geometrical structures present in the phase spaces of these systems have for our understanding of their dynamics. A second theme was the central importance of certain prototypical partial differential equations. These equations, which include the complex Ginzburg-Landau, nonlinear Schrödinger and Korteweg-de Vries equations, arise in many different contexts and hence have an importance that transcends their apparently special form. In this book, two sets of lectures explore this phenomenon in greater detail for the complex Ginzburg-Landau equation, one examining in detail the sorts of phenomena that can arise in this equation, and the other focussed on showing rigorously how a knowledge of the behavior of the solutions of the Ginzburg-Landau equation implies information about a host of more complicated systems. In addition to their ubiquity, the nonlinear Schrödinger and Korteweg-de Vries equation share the additional remarkable property of being completely integrable. The meaning and consequences of complete integrability are explored in the lectures of section 2. Finally, the last set of lectures looks specifically at problems in fluid mechanics and turbulence. More specifically, it examines the extent to which one can determine the limits of popular physically motivated heuristic theories of fluids like the renormalization group and the Kolmogorov scaling law.

Percy Deift
C. David Levermore
C. Eugene Wayne

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# Dynamical Systems and PDE's

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# An Introduction to KAM Theory

C. EUGENE WAYNE

## 1. Introduction

Over the past thirty years, the Kolmogorov-Arnold-Moser (KAM) theory has played an important role in increasing our understanding of the behavior of non-integrable Hamiltonian systems. I hope to illustrate in these lectures that the central ideas of the theory are, in fact, quite simple. With this in mind, I will concentrate on two examples and will forego generality for concreteness and (I hope) clarity. The results and methods which I will present are well-known to experts in the field but I hope that by collecting and presenting them in as simple a context as possible I can make them somewhat more approachable to newcomers than they are often considered to be.

The outline of the lectures is as follows. After a short historical introduction, I will explain in detail one of the simplest situations where the KAM techniques are used – the case of diffeomorphisms of a circle. I will then go on to discuss the theory in its original context, that of nearly-integrable Hamiltonian systems.

The problem which the KAM theory was developed to solve first arose in celestial mechanics. More than 300 years ago, Newton wrote down the differential equations satisfied by a system of massive bodies interacting through gravitational forces. If there are only two bodies, these equations can be explicitly solved and one finds that the bodies revolve on Keplerian ellipses about their center of mass. If one considers a third body (the "three-body-problem"), no exact solution exists – even if, as in the solar system, two of the bodies are much lighter then the third. In this case, however, one observes that the mutual gravitational force between these two "planets" is much weaker than that between either planet and the sun. Under these circumstances one can try to solve the problem perturbatively, first ignoring the interactions between the planets. This gives an **integrable** system, or one which can be solved explicitly, with

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each planet revolving around the sun oblivious of the other's existence. One can then try to systematically include the interaction between the planets in a perturbative fashion. Physicists and astronomers used this method extensively throughout the nineteenth century, developing series expansions for the solutions of these equations in the small parameter represented by the ratio of the mass of the planet to the mass of the sun. However, the convergence of these series was never established – not even when the King of Sweden offered a very substantial prize to anyone who succeeded in doing so. The difficulty in establishing the convergence of these series comes from the fact that the terms in the series have **small denominators** which we shall consider in some detail later in these lectures. One can obtain some physical insight into the origin of these convergence problems in the following way. As one learns in an elementary course in differential equations, a harmonic oscillator has a certain natural frequency at which it oscillates. If one subjects such an oscillator to an external force of the same frequency as the natural frequency of the oscillator, one has **resonance** effects and the motion of the oscillator becomes unbounded. Indeed, if one has a typical nonlinear oscillator, then whenever the perturbing force has a frequency that is a rational multiple of the natural frequency of the oscillator, one will have resonances, because the nonlinearity will generate oscillations of all multiples of the basic driving frequency.

In a similar way, one planet exerts a periodic force on the motion of a second, and if the orbital periods of the two are commensurate, this can lead to resonance and instability. Even if the two periods are not exactly commensurate, but only approximately so the effects lead to convergence problems in the perturbation theory.

It was not until 1954 that A. N. Kolmogorov [**8**] in an address to the ICM in Amsterdam suggested a way in which these problems could be overcome. His suggestions contained two ideas which are central to all applications of the KAM techniques. These two basic ideas are:

- Linearize the problem about an approximate solution and solve the linearized problem – it is at this point that one must deal with the small denominators.
- Inductively improve the approximate solution by using the solution of the linearized problem as the basis of a Newton's method argument.

These ideas were then fleshed out, extended, and applied in numerous other contexts by V. Arnold and J. Moser, ([**1**], [**9**]) over the next ten years or so, leading to what we now know as the KAM theory.

As I said above, we will consider the details of this procedure in two cases. The first, the problem of showing that diffeomorphisms of a circle are conjugate to rotations, was chosen for its simplicity – the main ideas are visible with fewer technical difficulties than appear in other applications. We will then look at the KAM theory in its original setting of small perturbations of integrable

Hamiltonian systems. I'll attempt to parallel the discussion of the case of circle diffeomorphisms as closely as possible in order to keep our focus on the main ideas of the theory and ignore as much as possible the additional technical complications which arise in this context.

**Acknowledgments:** It is a pleasure to thank Percy Deift and Andrew Török for many helpful comments about these notes.

## 2. Circle Diffeomorphisms

Let us begin by discussing one of the simplest examples in which one encounters small denominators, and for which the KAM theory provides a solution. It may not be apparent for the moment what this problem has to do with the problems of celestial mechanics discussed in the introduction, but almost all of the difficulties encountered in that problem also appear in this context but in ways which are less obscured by technical difficulties – this is, if you like, our warm-up exercise.

We will consider orientation preserving diffeomorphisms of the circle, or equivalently, their lifts to the real line:

$$\phi : R^1 \to R^1$$

$$\phi(x) = x + \tilde{\eta}(x) \text{ with } \tilde{\eta}(x+1) = \tilde{\eta}(x) \text{ and } \tilde{\eta}'(x) > -1 \ .$$

We wish to consider $\phi$ as a dynamical system, and study the behavior of its "orbits" – *i.e.* we want to understand the behavior of the sequences of points $\{\phi^{(n)}(x)(\text{mod}1)\}_{n=0}^{\infty}$, where $\phi^{(n)}$ means the $n$-fold composition of $\phi$ with itself. Typical questions of interest are whether or not these orbits are periodic, or dense in the circle.

The simplest such diffeomorphism is a rotation $R_\alpha(x) = x + \alpha$. Note that we understand "everything" about its dynamics. For instance, if $\alpha$ is rational, all the orbits of $R_\alpha$ are periodic, and none are dense. However, we would like to study more complicated dynamical systems than this. Thus we will suppose that

$$\phi(x) = x + \alpha + \eta(x) \ , \tag{1}$$

where as before, $\eta(x+1) = \eta(x)$ and $\eta'(x) > -1$. As I said in the introduction, I will not attempt to consider the most general case, but rather will focus on simplicity of exposition. Thus I will consider only **analytic** diffeomorphisms. Define the strips $S_\sigma = \{z \in C \mid |Imz| < \sigma\}$. Then I will assume that

$$\begin{aligned} \eta \in B_\sigma =&\{\eta \mid \eta(z) \text{ is analytic on } S_\sigma, \\ &\quad \eta(x+1) = \eta(x) \text{ and } \sup_{|Imz|<\sigma} |\eta(z)| \equiv \|\eta\|_\sigma < \infty\} \ . \end{aligned}$$

Note that one can assume that $\sigma < 1$, without loss of generality.

Our goal in this section will be to understand the dynamics of $\phi(x) = x + \alpha + \eta(x)$ when $\eta$ has small norm. One way to do this is to show that the dynamics of $\phi$ are "like" the dynamics of a system we understand – for instance, suppose that we could find a change of variables which transformed $\phi$ into a pure rotation. Then since we understand the dynamics of the rotation, we would also understand those of $\phi$. If we express this change of variables as $x = H(\xi)$, where $H(\xi+1) = 1 + H(\xi)$ preserves the periodicity of $\phi$, then we want to find $H$ such that

$$H^{-1} \circ \phi \circ H(\xi) = R_\rho(\xi) \ ,$$

or equivalently

$$\phi \circ H(\xi) = H \circ R_\rho(\xi) \ . \tag{2}$$

Such a change of variables is said to conjugate $\phi$ to the rotation $R_\rho$.

REMARK 2.1. *The relationship between this problem and the celestial mechanics questions discussed in the introduction now becomes more clear. In that case we wanted to understand the extent to which the motion of the solar system when we included the effects of the gravitational interaction between the various planets was "like" that of the simple Kepler system.*

In order to answer this question we need to introduce an important characteristic of circle diffeomorphisms, the **rotation number**

DEFINITION 2.1. *The rotation number of $\phi$ is*

$$\rho(\phi) = \lim_{n\to\infty} \frac{\phi^{(n)}(x) - x}{n} \ .$$

REMARK 2.2. *It is a standard result of dynamical systems theory that for any homeomorphism of the circle the limit on the right hand side of this equation exists and is independent of $x$. (See* [**6**], *p. 296.)*

REMARK 2.3. *Note that from the definition of the rotation number, it follows immediately that for any homeomorphism $H$, the map $\tilde{\phi} = H^{-1} \circ \phi \circ H$ has the same rotation number as $\phi$. (Since $\tilde{\phi}^{(n)} = H^{-1} \circ \phi^{(n)} \circ H$, and the initial and final factors of $H$ and $H^{-1}$ have no effect on the limit.)*

As a final remark about the rotation number we note that if $\phi(x) = x + \alpha + \eta(x)$, then an easy induction argument shows that $\rho(\phi) = \alpha + \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \eta \circ \phi^{(j)}(x)$. In particular, if $\alpha = \rho$, we have $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \eta \circ \phi^{(j)}(x) = 0$, so we have proved:

LEMMA 2.1. *If $\phi(x) = x + \rho + \eta(x)$ has rotation number $\rho$, then there exists some $x_0$ such that $\eta(x_0) = 0$.*

We must next ask about the properties we wish the change of variables $H$ to have. If we only demand that $H$ be a homeomorphism, then **Denjoy's Theorem** ([**6**] p. 301) says that if the rotation number of $\phi$ is irrational, we can always find an $H$ which conjugates $\phi$ to a rotation. However, if we want more detailed information about the dynamics it makes sense to ask that $H$ have additional smoothness. In fact, it is natural to ask that $H$ be as smooth as the diffeomorphism itself – in this case, analytic. (There will, in general, be some loss of smoothness even in this case. We will find, for example, that while there exists an analytic conjugacy function, $H$, its domain of analyticity will be somewhat smaller than that of $\phi$.) Surprisingly, the techniques which Denjoy used fail completely in this case, and the answer was not known until the late fifties when Arnold applied KAM techniques to answer the question in the case when

$\eta$ is small. Even more surprisingly, in order to even state Arnold's theorem, we have to discuss a little number theory.

Any irrational number can be approximated arbitrarily well by rational numbers, and in fact, **Dirichlet's Theorem** even gives us an estimate of how good this approximation is. More precisely, it says that given any irrational number $\rho$, there exist infinitely many pairs of integers $(m, n)$ such that $|\rho - (m/n)| < 1/n^2$. On the other hand, most irrational numbers can't be approximated much better than this.

DEFINITION 2.2. *The real number $\rho$ is of type $(K, \nu)$ if there exist positive numbers $K$ and $\nu$ such that $|\rho - (m/n)| > K|n|^{-\nu}$, for all pairs of integers $(m, n)$.*

PROPOSITION 2.1. *For every $\nu > 2$, almost every irrational number $\rho$ is of type $(K, \nu)$ for some $K > 0$.*

**Proof:** The proof is not difficult, but would take us a bit out of our way. The details can be found in [**3**], page 116, for example. Note also, that we can assume without loss of generality that $K \leq 1$, since if $\rho$ is of type $(\tilde{K}, \nu)$ for some $\tilde{K} > 1$, it is also of type $(1, \nu)$.

THEOREM 2.1 (ARNOLD'S THEOREM [**1**]). *Suppose that $\rho$ is of type $(K, \nu)$. There exists $\epsilon(K, \nu, \sigma) > 0$ such that if $\phi(x) = x + \rho + \eta(x)$ has rotation number $\rho$, and $\|\eta\|_\sigma < \epsilon(K, \nu, \sigma)$, then there exists an analytic and invertible change of variables $H(x)$ which conjugates $\phi$ to $R_\rho$.*

As mentioned above, Arnold's proof of this theorem used the KAM theory. The proof can be broken into two main parts – an analysis of a linearized equation, and a Newton's method iteration step. These same two steps will reappear in the next section when we discuss nearly integrable Hamiltonian systems, and they are characteristic of almost all applications of the KAM theory.

REMARK 2.4. *It may seem that by assuming that the diffeomorphism is of the form $\phi(x) = x + \rho + \eta(x)$, where $\rho$ is the rotation number of $\phi$, we are considering a less general situation than that described above in which we allowed $\phi$ to have the form $x + \alpha + \eta(x)$. As we shall see below, there is no real loss of generality in this restriction.*

### Step 1: Analysis of the Linearized equation

Note that since $\|\eta\|_\sigma$ is small, the diffeomorphism $\phi$ is "close" to the pure rotation $R_\rho$. Thus, we might hope that if a change of variables $H$ which satisfies (2) exists is would be close to the identity *i.e.* $H(x) = x + h(x)$, where $h$ is