The Nonlinear Limit–Point/Limit–Circle Problem

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To Ivana, Ondřej, and Frances
without whose support and encouragement
this would have forever remained a work in progress.

Preface

The purpose of this book is to present some new developments in the asymptotic analysis of nonlinear differential equations with particular attention paid to the limit–point/limit–circle problem. Nearly one hundred years have passed since Hermann Weyl first investigated this problem for second order linear differential equations. Since then this problem has been extended in various directions including:

- spectral analysis of differential operators of order $2n$;
- LP/LC problem for second order nonlinear differential equations;
- LP/LC problem for higher order nonlinear differential equations;
- the problem of existence of solutions not in L^2 for second order equations;
- the problem of existence of at least one solution in L^2 for second and higher order equations;
- the problem of existence of at least one solution in L^p for second and higher order equations.

Our attention here is focused on the extension of the classical Weyl problem to nonlinear equations in the sense that either all solutions are of the "nonlinear limit–circle type" or there is at least one solution that does not have this property. Some related problems, such as the existence of an L^2 solution, are not treated here. We should emphasize that for nonlinear problems, the existence of continuable solutions and singular solutions plays an important role. (This is discussed in detail in Chapter 2.)

The book consists of nine chapters. Chapter 1 discusses the origin of the limit–point/limit–circle problem including the motivation for the choice of this terminology. Chapter 2 gives the basic definitions and extension for nonlinear differential equations and examines the question of the existence of both continuable and singular solutions. Chapter 3 presents our results for second order nonlinear equations, including some necessary and sufficient conditions for a second order nonlinear equation to be of the "nonlinear limit–circle type." Chapter 4 describes some early attempts at obtaining limit–point type results for second and higher order nonlinear equations. In the last section in this chapter, we also describe some recently obtained results that are related to these earlier ones. In Chapter 5, we examine the connection between the limit–circle property and other properties of solutions of linear and nonlinear equations such as boundedness, oscillation, and convergence to zero. Chapters 6 and 7 examine the limit–point/limit–circle problem for third and fourth order equations, respectively. Chapter 8 is devoted to equations of arbitrary order, and Chapter 9 discusses the relationship between the limit–point/limit–circle problem and the spectral theory of differential operators. There are more than 120 references, and a number of open problems for future research are included.

Our joint interest in this problem began in the fall of 1993 when J. R. Graef visited Brno and gave a survey lecture on the status of the nonlinear limit–point/limit–circle problem. His own interest in the problem began in the late 1970s and included some collaboration with P. W. Spikes on second order nonlinear equations while they were both on the faculty at Mississippi State University. With that initial visit to Brno, the present authors began collaborating on the nonlinear limit–point/limit–circle problem and this led quite naturally to the present monograph.

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Basic Notation

$\mathbb{R}$	the set of real numbers;
$\mathbb{C}$	the set of complex numbers;
$\mathbb{R}_+$	the set of nonnegative real numbers, i.e., the interval $[0, \infty)$;
$\mathbb{R}^n$	the set $\mathbb{R} \times \cdots \times \mathbb{R}$ (n times);
L^2	the set of Lebesgue square integrable functions $u\colon \mathbb{R}_+ \to \mathbb{R}$;
$L([a, b])$	the set of Lebesgue integrable functions $u\colon [a, b] \to \mathbb{R}$;
$L_{loc}(\mathbb{R}_+)$	the set of integrable functions $u\colon \mathbb{R}_+ \to \mathbb{R}$ whose restriction to any interval $[a, b]$ belongs to $L([a, b])$;
$L^2_{loc}(\mathbb{R}_+)$	the set of locally square integrable functions on $[0, \infty)$;
$AC_{loc}(\mathbb{R}_+)$	set of all locally absolutely continuous functions on $[0, \infty)$;
$C^k([a, b])$	the set of k times continuously differentiable functions $u\colon [a, b] \to \mathbb{R}$;
$C^0(\mathbb{R}_+)$	the set of continuous functions $u\colon \mathbb{R}_+ \to \mathbb{R}$;
$g(t) = \mathcal{O}(h(t))$	$g(t)/h(t)$ is bounded as $t \to \infty$;
$[\![n]\!]$	the greatest integer function of n.

Contents

Chapter 1

Origins of the Limit–Point/ Limit–Circle Problem

In this chapter, we begin with a discussion of the origins of the limit–point/limit–circle problem including a motivation for the choice of this terminology. We then discuss its relationship to the notion of the deficiency index and describe the classical results for second order linear equations.

1.1. The Weyl Alternative

In 1910, Hermann Weyl [114] published his now classic paper on eigenvalue problems for second order linear differential equations of the form

$$(a(t)y')' + r(t)y = \lambda y, \quad t \in [0, \infty), \ \lambda \in \mathbb{C}, \tag{1.1}$$

and he classified this equation to be of the *limit–circle* type if every solution is square integrable, i.e., belongs to L^2, and to be of the *limit–point* type if at least one solution does not belong to L^2. In the ensuing years there has been a great deal of interest in the limit–point/limit–circle problem due to its importance in relation to the solution of certain boundary value problems (see Titchmarsh [109, 110]). As we will see later, the study of the analogous problem for nonlinear equations is relatively new and not nearly as extensive as for the linear case.

To understand the basis for Weyl's terminology, we begin with one of his fundamental results. The terminology limit–point/limit–circle arises in a somewhat natural way from the proof of this result, a sketch of which will be given.

Theorem 1.1. *If Im* $\lambda \neq 0$, *then* (1.1) *always has a solution* $y \in L^2(\mathbb{R}_+)$, *i.e.,*

$$\int_0^\infty |y(t)|^2 dt < \infty.$$

Sketch of the Proof. For λ with $\mathrm{Im}\,\lambda \neq 0$, let φ and ψ be two linearly independent solutions of (1.1) satisfying the initial conditions

$$\varphi(0,\lambda) = 1, \quad \psi(0,\lambda) = 0,$$
$$\varphi'(0,\lambda) = 0, \quad \psi'(0,\lambda) = 1.$$

The functions $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are analytic in λ on $\mathbb{C}$. Then, any other solution y is a linear combination of these solutions, say,

$$y(t,\lambda) = \varphi(t,\lambda) + m(\lambda)\psi(t,\lambda).$$

Choose $b > 0$ and let c_1 and c_2 be arbitrary but fixed constants; we want to determine $m(\lambda)$ so that the solution y satisfies

$$c_1 y(b,\lambda) + c_2 y'(b,\lambda) = 0. \tag{1.2}$$

This desired value of m depends on λ, b, c_1, and c_2, and in fact has the form of the linear fractional transformation

$$m = \frac{Az + B}{Cz + D}.$$

The image of the real axis in the *z-plane* is a circle $\mathcal{C}_b$ in the *m-plane*. The solution y will satisfy (1.2) if and only if m is on $\mathcal{C}_b$. An argument using Green's identity shows that this is true if and only if

$$\int_0^b |y(s)|^2 ds = \frac{Im\, m}{Im\, \lambda},$$

and the radius of the circle $\mathcal{C}_b$ is

$$r_b = \left(2Im\,\lambda \int_0^b |y(s)|^2 ds\right)^{-1} \tag{1.3}$$

Observe that if $b_1 < b$, then

$$\int_0^{b_1} |y(s)|^2 ds < \int_0^b |y(s)|^2 ds,$$

so $r_b < r_{b_1}$, i.e., the circle $\mathcal{C}_{b_1}$ contains the circle $\mathcal{C}_b$ in its interior. Thus, as $b \to \infty$, the circles $\mathcal{C}_b$ converge either to a circle $\mathcal{C}_\infty$ or to a point m_∞. If the limiting form is a circle, then $r_\infty > 0$, and so (1.3) implies

$$\int_0^\infty |y(s)|^2 ds < \infty,$$

i.e., $y \in L^2$ for any m on $\mathcal{C}_\infty$. If the limit is the point m_∞, then $r_\infty = 0$ and there is only one solution in L^2. □

Titchmarsh [109, 110] discusses the connection between the limit–point property and the existence of a unique Green's function for second order linear differential equations. In the limit–circle case, the Green's function depends on a parameter.

Essential to the study of the limit–point/limit–circle problem is the following result of Weyl.

Theorem 1.2. *If* (1.1) *is limit-circle for some* $\lambda_0 \in \mathbb{C}$, *then* (1.1) *is limit-circle for all* $\lambda \in \mathbb{C}$.

In particular, Theorem 1.2 holds for $\lambda = 0$, that is, if we can show that equation (1.1) is limit–circle for $\lambda = 0$, then it is limit–circle for all values of λ. Moreover, if (1.1) is not limit–circle for $\lambda = 0$, then it is not limit–circle for any value of λ. In view of Theorem 1.1, for second order equations the problem reduces to whether equation (1.1) with $\operatorname{Im}\lambda \neq 0$ has one (limit–point case) or two (limit–circle case) solutions in L^2 (this is known as the *Weyl Alternative*). As we will see later, the situation for higher order equations is somewhat different in that the limit–point and limit–circle cases do not form a dichotomy.

The limit–point/limit–circle problem then becomes that of determining necessary and/or sufficient conditions on the coefficient functions to be able to distinguish between these two cases. Weyl's results have spawned research in a variety of directions including the study of what is called the *deficiency index problem*, which we describe in the next section.

1.2. The Deficiency Index Problem

The extension of the limit–point/limit–circle problem for second order equations to equations of higher order leads to the study of the deficiency index for self-adjoint differential operators. Consider the differential expression

$$\ell(y) \equiv \sum_{i=0}^{n}(-1)^i\left(p_i(t)y^{(i)}\right)^{(i)} = (-1)^n(p_n(t)y^{(n)})^{(n)}$$
$$+ (-1)^{n-1}(p_{n-1}(t)y^{(n-1)})^{(n-1)} + \cdots + (-1)(p_1(t)y^{(1)})^{(1)} + p_0(t)y \quad (1.4)$$

where p_i, $i = 0, \ldots, n$ are real-valued functions for $t \in \mathbb{R}_+$, $p_n(t) > 0$, and p_n^{-1}, p_{n-1}, $\ldots$, $p_0 \in L_{\text{loc}}(\mathbb{R}_+)$. Then, the minimal operator L_0 associated with this differential expression has self-adjoint extensions; see, for example, Naimark

[94, §17]. The *deficiency index*, denoted by m, of L_0 is the number of linearly independent solutions of

$$\ell(y) = \lambda y, \qquad \operatorname{Im}\lambda \neq 0, \tag{1.5}$$

that are in L^2 (see Devinatz [34]). The number m is independent of λ (as long as $\operatorname{Im}\lambda \neq 0$) and the possible values for m are

$$n, n+1, \ldots, 2n,$$

with the exact value taken depending on the coefficients p_i.

There is a higher order counterpart of Theorem 1.2 above (see Naimark [94, Theorem 4, p. 93]). That is, (1.5) has all its solutions in L^2, i.e., is (higher order) limit-circle if and only if the equation

$$\ell(y) = 0 \tag{1.6}$$

has all its solutions in L^2. Thus, in what follows, we only study the limit–circle problem for equation (1.6).

For some time it was believed that the deficiency index was always either n or $2n$, and as we saw in Section 1.1, this is the case for second order equations. However, for higher order equations, any value between n and $2n$ is possible (see Glazman [57]). Some authors refer to the case $m = n$ as the limit–point case for higher order linear equations.

Several conjectures on the value of the deficiency index m have been posed over the years; here we will briefly discuss a couple of them.

• Everitt's conjecture (1961): *If $p_j \geq 0$ for all $j = 0, \ldots, n$, then $m = n$.*

• Kauffman (1976) disproved this conjecture giving the example that the operator $-(x^a y''')''' + Kx^{6-a}y$ has the deficiency index $m > 3$.

• McLeod's conjecture (1962): *If $p_j \geq 0$ for all $j = 0, \ldots, n$, then $n \leq m \leq 2n - 1$ and all m occur.*

As far as we know this conjecture is still open.

• Paris and Wood (1981) proved: *For every j with $0 \leq j \leq [\![\frac{n+1}{4}]\!]$ there exists a real formally self-adjoint expression of order $2n$ with nonnegative coefficients having deficiency index $m = n + 2j$.*

• Schultze (1992) improved the range of values covered by showing: *For every j, $0 \leq j < n/2$ there exists such an expression having deficiency index $m = n + 2j$.*

This still leaves half of the values between n and $2n$ unaccounted for.

An excellent historical account of the development of the deficiency index problem can be found in the survey article by Everitt [49] which contains more than sixty references to work prior to 1976; also see [50, 51].

1.3. Second Order Linear Equations

In the study of the linear equation

$$(a(t)y')' + r(t)y = 0, \tag{1.7}$$

where $a, r : \mathbb{R}_+ \to \mathbb{R}$ is continuous, $a', r' \in AC_{\text{loc}}(\mathbb{R}_+)$, $a'', r'' \in L^2_{\text{loc}}(\mathbb{R}_+)$, $a(t) > 0$, and $r(t) > 0$, it has proven useful (see Dunford and Schwartz [37] or Burton and Patula [24]) to make the transformation

$$s = \int_0^t \left[\frac{r(u)}{a(u)}\right]^{\frac{1}{2}} du, \quad x(s) = y(t), \tag{1.8}$$

and let "$\cdot$" denote $\frac{d}{ds}$. Then, we have

$$y'(t) = \dot{x}(s)\frac{ds}{dt} = [r(t)/a(t)]^{\frac{1}{2}}\dot{x}(s),$$

$$a(t)y'(t) = [a(t)r(t)]^{\frac{1}{2}}\dot{x}(s),$$

and

$$\begin{aligned}(a(t)y'(t))' &= [a(t)r(t)]^{\frac{1}{2}}\ddot{x}(s)[r(t)/a(t)]^{\frac{1}{2}} + \frac{1}{2}[a(t)r(t)]^{-\frac{1}{2}}[a(t)r(t)]'\dot{x}(s)\\ &= r(t)\ddot{x}(s) + [a(t)r(t)]'\dot{x}(s)/2[a(t)r(t)]^{\frac{1}{2}}.\end{aligned}$$

Equation (1.7) then becomes

$$\ddot{x}(s) + 2p(t)\dot{x}(s) + x(s) = 0$$

where

$$p(t) = [a(t)r(t)]'/4a^{\frac{1}{2}}(t)r^{\frac{3}{2}}(t).$$

The following theorem due to Dunford and Schwartz [37, p. 1410] is probably the best known limit–circle result for equation (1.7).

Theorem 1.3. *Assume that*

$$\int_0^\infty \left|\left[\frac{(a(u)r(u))'}{a^{\frac{1}{2}}(u)r^{\frac{3}{2}}(u)}\right]' + \frac{\{[a(u)r(u)]'\}^2}{4a^{\frac{3}{2}}(u)r^{\frac{5}{2}}(u)}\right| du < \infty. \tag{1.9}$$

If

$$\int_0^\infty [1/(a(u)r(u))^{\frac{1}{2}}]du < \infty, \tag{1.10}$$

then equation (1.7) *is limit–circle, i.e., every solution* $y(t)$ *of* (1.7) *satisfies*

$$\int_0^\infty y^2(u)du < \infty.$$

Their corresponding limit–point result is the following.

Theorem 1.4. *Assume that* (1.9) *holds. If*

$$\int_0^\infty [1/(a(u)r(u))^{\frac{1}{2}}]du = \infty, \tag{1.11}$$

then equation (1.7) *is limit–point, i.e., there is a solution* $y(t)$ *of* (1.7) *such that*

$$\int_0^\infty y^2(u)du = \infty.$$

Remark 1.1. Everitt [47] proved that the linear equation (1.7) is of the limit–circle type if (1.10) holds and the condition (1.9) of Dunford and Schwartz is replaced by

$$\int_0^\infty \{[a(u)(a(u)r(u))^{-\frac{5}{4}}(a(u)r(u))']'\}^2 du < \infty. \tag{1.12}$$

Wong [118, Proposition, p. 424] showed that equation (1.7) is of the limit–circle type if (1.10) holds and

$$\int_0^\infty \left|[a(u)r(u)]^{-\frac{1}{4}}\{a(u)[(a(u)r(u))^{-\frac{1}{4}}]'\}'\right| du < \infty. \tag{1.13}$$

By showing that conditions (1.10) and (1.12) imply (1.13), he thus has an extension of Everitt's result.

Remark 1.2. When $a(t) \equiv 1$ so that equation (1.7) reduces to

$$y'' + r(t)y = 0, \tag{1.14}$$

then condition (1.9) of Dunford and Schwartz becomes

$$\int_0^\infty \left|\frac{r''(u)}{r^{3/2}(u)} - \frac{5}{4}\frac{[r'(u)]^2}{r^{5/2}(u)}\right| du < \infty. \tag{1.15}$$

Burton and Patula [24, Theorem 1] (also see Knowles [84, Theorem 5]) proved a variation of Theorem 1.3 for equation (1.14) by replacing conditions (1.9) and (1.10) (with $a(t) \equiv 1$) of Dunford and Schwartz with the single condition

$$\int_0^\infty \left[\frac{r(0)}{r(s)}\right]^{1/2} \exp\left\{(1/4)\int_0^s \left|\frac{(r'(u))^2}{4r^{\frac{5}{2}}(u)} + \left[\frac{r'(u)}{r^{\frac{3}{2}}(u)}\right]'\right| du\right\} ds < \infty. \qquad (1.16)$$

Knowles [82] showed that conditions like (1.9) with (1.10), or (1.12) with (1.10), are special cases of a broader class of conditions for linear equations to be of the limit–circle type. For example, Knowles shows that a result of Pavljuk [100], namely, equation (1.14) is of the limit–circle type provided (1.10) holds and

$$\left|\frac{r''(t)}{r(t)} - \frac{5}{4}\left(\frac{r'(t)}{r(t)}\right)^2\right| \quad \text{is bounded,} \qquad (1.17)$$

is also in this family of conditions.

Ráb [101, Section 2.3] obtained some asymptotic formulas for solutions of equation (1.14) under the assumption that

$$\int_0^\infty \left|\frac{r''(u)}{r^{3/2}(u)} - \eta\frac{[r'(u)]^2}{r^{5/2}(u)}\right| du < \infty \qquad (1.18)$$

with $\eta \in \mathbb{R} - \{\frac{3}{2}\}$.

Harris [71] also studied the limit–circle problem for equation (1.7) under conditions similar to those described above.

Titchmarsh [109, Theorem 5.11] or [110, Section 3] proved the following limit–circle result.

Theorem 1.5. *If* $r' > 0$, $r(t) \to +\infty$ *as* $t \to \infty$, r'' *is eventually of one sign,* $r'(t) = \mathcal{O}(|r(t)|^c)$ *as* $t \to \infty$ *for* $0 < c < 3/2$, *and*

$$\int_0^\infty \frac{1}{r^{\frac{1}{2}}(u)} du < \infty,$$

then equation (1.14) *is of the limit–circle case.*

The following lemma due to Coppel [31] provides some insight into the relationship between conditions such as (1.9), (1.12), (1.13), (1.15), (1.16), and (1.18).