Continuous Symmetries, Lie Algebras, Differential Equations and Computer Algebra

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Preface

The purpose of this book is to provide a comprehensive introduction to the application of continuous symmetries and their Lie algebras to ordinary and partial differential equations. The study of symmetries of differential equations provides important information about the behaviour of differential equations. The symmetries can be used to find exact solutions. They can be applied to verify and develop numerical schemes. One can also obtain conservation laws of a given differential equation with the help of the continuous symmetries. Gauge theory is also based on the continuous symmetries of certain relativistic field equations.

Apart from the standard techniques in the study of continuous symmetries, the book includes: the Painlevé test and symmetries, invertible point transformation and symmetries, Lie algebra valued differential forms, gauge theory, Yang-Mills theory and chaos, self-dual Yang-Mills equation and soliton equations, Bäcklund transformation, Lax representation, Bose operators and symmetries, discrete systems and invariants.

Each chapter includes computer algebra applications. Examples are the finding of the determining equation for the Lie symmetries, finding the curvature for a given metric tensor field and calculating the Killing vector fields for a metric tensor field.

The book is suitable for use by students and research workers whose main interest lies in finding solutions of differential equations. It therefore caters for readers primarily interested in applied mathematics and physics rather than pure mathematics. The book provides an application orientated text that is reasonably self-contained. A large number of worked examples have been included in the text to help the readers working independently of a teacher. The advance of algebraic computation has made it possible to write programs for the tedious calculations in this research field. Thus the last chapter gives a survey on computer algebra packages.

End of proofs are indicated by ♠. End of examples are indicated by ♣.

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Any useful suggestions and comments are welcome.

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Notation

$\emptyset$	empty set
$\mathbf{N}$	natural numbers
$\mathbf{Z}$	integers
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
$\mathbf{R}^+$	nonnegative real numbers
$\mathbf{C}$	complex numbers
$\mathbf{R}^n$	n-dimensional Euclidian space
$\mathbf{C}^n$	n-dimensional complex linear space
i	$:= \sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\mathbf{x} \in \mathbf{R}^n$	element $\mathbf{x}$ of $\mathbf{R}^n$
$A \subset B$	subset A of set B
$A \cap B$	the intersection of the sets A and B
$A \cup B$	the union of the sets A and B
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
u	dependent variable
t	independent variable (time variable)
x	independent variable (space variable)
$\mathbf{x}^T = (x_1, x_2, \ldots, x_m)$	vector of independent variables, ^{T} means transpose
$\mathbf{u}^T = (u_1, u_2, \ldots, u_n)$	vector of dependent variables, ^{T} means transpose
$\lVert \cdot \rVert$	norm
$\mathbf{x} \cdot \mathbf{y}$	scalar product (inner product)
$\mathbf{x} \times \mathbf{y}$	vector product
$\otimes$	Kronecker product, tensor product
det	determinant of a square matrix
tr	trace of a square matrix
I	unit matrix
$[\,,\,]$	commutator
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
d	exterior derivative
λ	eigenvalue
ϵ	real parameter
$\wedge$	Grassmann product (exterior product, wedge product)
H	Hamilton function
L	Lagrange function

Contents

Chapter 1

Introduction

Sophus Lie (1842–1899) and Felix Klein (1849–1925) studied mathematical systems from the perspective of those transformation groups which left the systems invariant. Klein, in his famous "Erlanger" program, pursued the role of finite groups in the studies of regular bodies and the theory of algebraic equations, while Lie developed his notion of continuous transformation groups and their role in the theory of differential equations. Today the theory of continuous groups is a fundamental tool in such diverse areas as analysis, differential geometry, number theory, atomic structure and high-energy physics. In this book we deal with Lie's theorems and extensions thereof, namely its applications to the theory of differential equations.

It is well known that many, if not all, of the fundamental equations of physics are nonlinear and that linearity is achieved as an approximation. One of the important developments in applied mathematics and theoretical physics over the recent years is that many nonlinear equations, and hence many nonlinear phenomena, can be treated as they are, without approximations, and be solved by essentially linear techniques.

One of the standard techniques for solving linear partial differential equations is the Fourier transform. During the past 25 years it was shown that a class of physically interesting nonlinear partial differential equations can be solved by a nonlinear extension of the Fourier technique, namely the inverse scattering transform. This reduces the solution of the Cauchy problem to a series of linear steps. This method, originally applied to the Korteweg-de Vries equation, is now known to be applicable to a large class of nonlinear evolution equations in one space and one time variable, to quite a few equations in $2 + 1$ dimensions and also to some equations in higher dimensions.

Continuous group theory, Lie algebras and differential geometry play an important role in the understanding of the structure of nonlinear partial differential equations, in particular for generating integrable equations, finding Lax pairs, recursion operators, Bäcklund transformations and finding exact analytic solutions.

Most nonlinear equations are not integrable and cannot be treated via the inverse scattering transform, nor its generalizations. They can of course be treated by numerical methods, which are the most common procedures. Interesting qualitative and quantitative features are however often missed in this manner and it is of great value to be able to obtain, at least, particular exact analytic solutions of nonintegrable equations. Here group theory and Lie algebras play an important role. Indeed, Lie group theory was originally created as a tool for solving ordinary and partial differential equations, be they linear or nonlinear.

New developments have also occurred in this area. Some of them have their origins in computer science. The advent of algebraic computing and the use of such computer languages for symbolic computations such as REDUCE, MACSYMA, AXIOM, MAPLE, MATHEMATICA, SYMBOLICC++ etc., have made it possible (in principle) to write computer programs that construct the Lie algebra of the symmetry group of a differential equation. Other important advances concern the theory of infinite dimensional Lie algebras, such as loop algebras, Kac-Moody and Virasoro algebras which frequently occur as Lie algebras of the symmetry groups of integrable equations in $2 + 1$ dimensions such as the Kadomtsev-Petviashvili equation. Furthermore, practical and computerizable algorithms have been proposed for finding all subgroups of a given Lie group and for recognizing Lie algebras given their structure constants.

In chapter 2 we give an introduction into group theory. Both finite and infinite groups are discussed.

Lie and Lie transformation groups are introduced in chapter 3. In particular, the classical Lie groups are studied in detail.

Chapter 4 is devoted to the infinitesimal transformations (vector fields) of Lie transformation groups. In particular, the three theorems of Lie are discussed.

Chapter 5 gives a comprehensive introduction into Lie algebras. We also discuss representations of Lie algebras in details. Many examples are provided to clarify the definitions and theorems.

The form-invariance of partial differential equations under Lie transformation groups is illustrated by way of examples in chapter 6. This should be seen as an introduction to the development of the theory of invariance of differential equations by the jet bundle formalism. The Gauge transformation for the Schrödinger equation is also discussed. We also show how the electromagnetic field A_μ is coupled to the wave function ψ.

Chapter 7 deals with differential geometry. Theorems and definitions (with examples) are provided that are of importance in the application of Lie algebras to differential equations. A comprehensive introduction into differential forms and tensor fields is given.

The Lie derivative is of central importance for continuous symmetries. In chapter 8 we study invariance and conformal invariance of geometrical objects, i.e. functions, vector fields, differential forms, tensor fields, etc..

In chapter 9 the jet bundle formalism in connection with the prolongation of vector fields and (partial) differential equations is studied. The application of the Lie derivative in the jet bundle formalism is analysed to obtain the invariant Lie algebra. Explicit analytic solutions are then constructed by applying the invariant Lie algebra. These are the so-called similarity solutions which are of great theoretical and practical importance. The direct method is also introduced.

In chapter 10 the generalisation of the Lie point symmetry vector fields is considered. These generalised vector fields are known as the Lie-Bäcklund symmetry vector fields. Similarity solutions are constructed from the Lie-Bäcklund vector fields. The connection with gauge transformations is also discussed.

In chapter 11 the inverse problem is considered. This means that a partial differential equation is constructed from a given Lie algebra which is spanned by Lie point or Lie-Bäcklund symmetry vector fields.

A list of Lie symmetry vector fields of some important partial differential equations in physics is included in chapter 12. In particular the Lie symmetry vector fields for the Maxwell-Dirac equation have been calculated.

In chapter 13, the Gateaux derivative is defined. A Lie algebra is introduced using the Gateaux derivative. Furthermore, recursion operators are defined and applied. Then we can find hierarchies of integrable equations.

In chapter 14 we introduce Bäcklund transformations for partial and ordinary differential equations.

For soliton equations the Lax representations are the starting point for the inverse scattering method. In chapter 15 we discuss the Lax representation. Many illustrative examples are given.

The important concept of conservation laws is discussed in chapter 16. The connection between conservation laws and Lie symmetry vector fields is of particular interest. Extensive use is made of the definitions and theorems of exterior differential forms. The Cartan fundamental form plays an important role regarding the Lagrange density and Hamilton density.

In chapter 17 the Painlevé test is studied with regard to the symmetries of ordinary and partial differential equations. The Painlevé test provides an approach to study the integrability of ordinary and partial differential equations. This approach is studied and several examples are given. In particular a connection between the singularity manifold and similarity variables is presented.

In chapter 18 the extension of differential forms, discussed in chapter 7, to Lie algebra valued differential forms is studied. The covariant exterior derivative is defined. Then the Yang-Mills equations and self-dual Yang-Mills equations are introduced. It is conjectured that the self-dual Yang-Mills equations are the master equations of all integrable equations such as the Korteweg-de Vries equation.

The connection between nonlinear autonomous systems of ordinary differential equations, first integrals, Bose operators and Lie algebras is studied in chapter 19. It is shown that ordinary differential equations can be expressed with Bose operators. Then the time-evolution can be calculated using the Heisenberg picture. An extension to nonlinear partial differential equations is given where Bose field operators are considered.

Chapter 20 gives a survey of computer algebra packages. Of particular interest are the computer programs available for the calculation of symmetry vector fields.

The emphasis throughout this book is on differential equations that are of importance in physics and engineering. The examples and applications consist mainly of the following equations: the Korteweg-de Vries equation, the sine-Gordon equation, Burgers' equation, linear and nonlinear diffusion equations, the Schrödinger equation, the nonlinear Klein-Gordon equation, nonlinear Dirac equations, Yang-Mills equations, the Lorenz model, the Lotka-Volterra model and damped anharmonic oscillators.

Each chapter includes a section on computer algebra applications.

Chapter 2

Groups

2.1 Definitions and Examples

In this section we introduce some elementary definitions and fundamental concepts in general group theory. We present examples to illustrate these concepts and show how different structures form a group.

Let us define a group as an abstract mathematical entity Miller [84], Baumslag and Chandler [6].

Definition 2.1 *A* **group** *G is a set $e, g_1, g_2, \cdots \in G$ not necessarily countable, together with an operator, called group composition $(\cdot)$, such that*

1. *Closure: $g_i \in G, g_j \in G \;\Rightarrow\; g_i \cdot g_j \in G$.*
2. *Assosiativity: $g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k$.*
3. *Existence of identity $e \in G$: $e \cdot g_i = g_i = g_i \cdot e$ for all $g_i, e \in G$.*
4. *Existence of inverse $g_i^{-1} \in G$: $g_i \cdot g_i^{-1} = g_i^{-1} \cdot g_i = e$ for all $g_i \in G$.*
5. *A group that obeys a fifth postulate $g_i \cdot g_j = g_j \cdot g_i$ for all $g_i, g_j \in G$, in addition to the four listed above is called an* **abelian group** *or* **commutative group**.

The group composition in an abelian group is often written in the form $g_i + g_j$. The element $g_i + g_j$ is called the sum of g_i and g_j and G is called an **additive group**.

Definition 2.2 *If a group G consists of a finite number of elements, then G is called a* **finite group**; *otherwise, G is called an* **infinite group**.

Example: The set of integers **Z** with addition as group composition is an infinite additive group with $e = 0$. ♣

Example: The set $\{1, -1\}$ with multiplication as group composition is a finite abelian group with $e = 1$. ♣

Definition 2.3 *Let G be a finite group. The number of elements of G is called the dimension or* **order** *of G.*

Definition 2.4 *A nonempty subset H of G is called a* **subgroup** *of G if H is a group with respect to the composition of the group G. We write $H < G$.*

Hence a nonempty subset H is a subgroup of G if and only if $h_i^{-1} \cdot h_j \in H$ for any $h_i, h_j \in H$. For a family $\{H_\lambda\}$ of subgroups of G, the intersection $\bigcap_\lambda H_\lambda$ is also a subgroup.

Theorem 2.1 *The identity element e is unique.*

Proof: Suppose $e' \in G$ such that $e' \cdot g_i = g_i \cdot e' = e$ for all $g_i \in G$. Setting $g_i = e$, we find $e \cdot e' = e' \cdot e = e$. But $e' \cdot e = e'$ since e is an identity element. Therefore, $e' = e$. ♠

Theorem 2.2 *The inverse element g_i^{-1} of g_i is unique.*

Proof: Suppose $g_i' \in G$ such that $g_i \cdot g_i' = e$. Multiplying on the left by g_i^{-1} and using the assosiative law, we get $g_i^{-1} = g_i^{-1} \cdot e = g_i^{-1} \cdot (g_i \cdot g_i') = (g_i^{-1} \cdot g_i) \cdot g_i' = e \cdot g_i' = g_i'$. ♠

Theorem 2.3 *The order of a subgroup of a finite group divides the order of the group.*

This theorem is called **Lagrange's theorem**. For the proof we refer to the literature (Miller [84]).

Definition 2.5 *Let H be a subgroup of G and $g \in G$. The set*

$$Hg := \{ \, hg \; : \; h \in H \, \}$$

is called a right coset of H. The set

$$gH := \{ \, gh \; : \; h \in H \, \}$$

is called a left coset of H.

Definition 2.6 *A subgroup N of G is called* **normal (invariant, self-conjugate)** *if $gNg^{-1} = N$ for all $g \in G$.*

If N is a normal subgroup we can construct a group from the cosets of N, called the **factor group** G/N. The elements of G/N are the cosets gN, $g \in G$. Of course, two cosets gN, $g'N$ containing the same elements of G define the same element $G/N : gN = g'N$. Since N is normal it follows that

$$(g_1N)(g_2N) = (g_1N)(g_2N) = g_1Ng_2 = g_1g_2N$$

as sets. Note that $NN = N$ as sets.

Consider an element g_i of a finite group G. If g_i is of order d, where the **order** of a group element is the smallest positive integer d with $g_i^d = g_1$ (identity), then the different powers of g_i are $g_i^0\,(= g_1), g_i, g_i^2, \ldots g_i^{d-1}$. All the powers of g_i form a group $< g_i >$ which is a subgroup of G and is called a **cyclic group**. This is an abelian group where the order of the subgroup $< g_i >$ is the same as the order of the element g_i.

A way to partition G is by means of **conjugacy classes**.

Definition 2.7 *A group element h is said to be conjugate to the group element $k, h \sim k$, if there exists a $g \in G$ such that*

$$k = ghg^{-1}.$$

It is easy to show that conjugacy is an equivalence relation, i.e., (1) $h \sim h$ (reflexive), (2) $h \sim k$ implies $k \sim h$ (symmetric), and (3) $h \sim k, k \sim j$ implies $h \sim j$ (transitive). Thus, the elements of G can be divided into conjugacy classes of mutually conjugate elements. The class containing e consists of just one element since

$$geg^{-1} = e$$

for all $g \in G$. Different conjugacy classes do not necessarily contain the same number of elements.

Let G be an abelian group. Then each conjugacy class consists of one group element each, since

$$ghg^{-1} = h, \quad \text{for all} \quad g \in G.$$

Let us now give a number of examples to illustrate the definitions given above.

Example: A **field** is an (infinite) abelian group with respect to addition. The set of nonzero elements of a field forms a group with respect to multiplication, which is called a multiplicative group of the field. ♣

Example: A **linear vector space** over a field K (such as the real numbers $\mathbf{R}$) is an abelian group with respect to the usual addition of vectors. The group composition of two elements (vectors) $\mathbf{a}$ and $\mathbf{b}$ is their vector sum $\mathbf{a}+\mathbf{b}$. The identity is the zero vector and the inverse of an element is its negative. ♣

Example: Let N be an integer with $N \geq 1$. The set

$$\{ \, e^{2\pi i n/N} \quad : \; n = 0, 1, \ldots, N-1 \, \}$$

is an abelian (finite) group under multiplication since

$$e^{2\pi i n/N} e^{2\pi i m/N} = e^{2\pi i (n+m)/N}$$