ELEMENTARY DIFFERENTIAL EQUATIONS
WITH BOUNDARY VALUE PROBLEMS

DAVID L. POWERS

Elementary Differential Equations

with Boundary Value Problems

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Preface

This book was written for second-year engineering and science students. The intended audience and my own experience as an engineering student have influenced the writing in three ways. First, applications motivate and illustrate the mathematics throughout. Second, methods are presented before theory wherever possible, so that the student approaches generalizations with a body of examples in mind. Third, most theorems are not proved, although they are explained, illustrated, and interpreted.

Two semesters of calculus is the required background. Infinite series, for Chapters 4, 9, and 10, and partial derivatives, for Chapter 10 and a few scattered sections, can be studied concurrently. Except for the bare facts about determinants, no linear algebra is needed in the bulk of the book. However most of Chapter 7, Systems of Linear Differential Equations, assumes a working knowledge of matrices. Appendix A is a self-contained text on matrix algebra and systems of algebraic equations for review or class presentation.

To accommodate different courses, curricula, teachers, and students, I have made this book as flexible as possible. Chapters 1–3 are to be taken in order; 4–8 are independent of each other; 9 and 10 form a sequence, and both draw on 4. Within each chapter, the material is arranged to allow different stopping points. The *Instructor's Manual* contains the particulars of chapter and section dependence as well as other useful information. Topics were chosen with a view to the current and probable future demands of the various engineering and science disciplines. There is enough material to design a two-semester course with any one of several different biases—toward classical applications, for instance, or systems and control, or computation.

The book has a number of special features that enhance its value as a text and reference.

* Over 225 examples illustrate definitions and theorems (in both the positive and negative senses) and guide the student in the use of new methods.
* More than 1500 exercises are provided, ranging from drill to novel applications, extensions of methods and theory, and previews of future material.

* Solutions of odd-numbered exercises are in the back of the book. Answers to even-numbered exercises are available. Some 300 exercises are worked in detail in the *Student's Partial Solutions Manual* written to accompany this text.

* Miscellaneous exercises conclude each chapter. Some of these are drill exercises for test preparation. Other problems require several sections' results, develop new methods, or take old methods in new directions.

* Notes and references at the end of each chapter comment on the subject from a broader viewpoint, telling why and to whom it is important, how it is related to others and where to find out more about it. A bibliography is at the end of the book.

* Each chapter has at least one section on an advanced or unusual topic. For example, Section 3.6 states and proves some theorems on boundedness and oscillation; Section 8.6 is an introduction to Jacobian elliptic functions.

* Appendix B, Mathematical References, lists some useful formulas and theorems from trigonometry, algebra, and calculus.

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1 First-Order Equations

1.1 Introduction

In many important physical problems the development in time of a particular quantity is controlled by a fundamental physical law. A good example is provided by a chemical solution in a "stirred-tank chemical reactor." This is a tank containing a solution, initially at a particular concentration. When the period of observation begins, solution flows in continuously at a given rate and concentration, and the contents of the tank are drawn off continuously at a given rate. There is supposed to be a stirring device in the tank to ensure that the concentration of the solution is uniform throughout the tank at any time. It is usually required to find the amount (mass) of solute in the tank as a function of time. The equation governing this quantity is found by applying the law of conservation of mass in this form:

$$\text{accumulation rate} = \text{rate in} - \text{rate out}. \tag{1.1}$$

Example 1

A 200-liter tank is initially filled with brine (a solution of salt in water) at a concentration of 2 grams per liter. Then brine flows in at a rate of 8 liters per minute with a concentration of 4 grams per liter. The well-stirred contents of the tank are drawn off at a rate of 8 liters per minute. Express the law of conservation of mass for the salt in the tank.

Let $u(t)$ be the mass of salt in the tank, measured in grams. The rate at which salt enters is

$$\text{rate in} = \frac{8\ \text{liter}}{\text{min}} \times \frac{4\ \text{g}}{\text{liter}} = \frac{32\ \text{g}}{\text{min}}.$$

The rate at which salt leaves is

$$\text{rate out} = \frac{8\ \text{liter}}{\text{min}} \times \frac{u(t)\ \text{g}}{200\ \text{liter}} = \frac{0.04\ \text{g}}{\text{min}} u(t).$$

The rate at which salt accumulates in the tank is just du/dt, measured in grams per minute. Thus the mass balance is

accumulation rate = rate in − rate out

$$\frac{du}{dt} = \frac{32\text{ g}}{\text{min}} - \frac{0.04u(t)\text{ g}}{\text{min}}.$$

The units of measurement are included as a check on consistency. They are usually dropped at this stage, and the mass balance equation is written

$$\frac{du}{dt} = -0.04u + 32, \qquad 0 < t. \tag{1.2}$$

The inequality, $0 < t$, reminds us that the equation is valid after the experiment starts.

Many variants are possible in problems of this type: the inflow and outflow rates might be different or nonconstant, a chemical reaction might take place in the tank, the solution might become saturated, and so on. But in any event the accumulation rate term will cause the derivative of the unknown quantity u to appear in the mass balance equation. This brings us to the subject of our study.

Definition 1.1 A relationship between a function and its derivatives is called a *differential equation.* The highest-order derivative that appears is called the *order* of the differential equation.

The mass balance equation of Example 1, Eq. (1.2), is a first-order differential equation. We shall see many more examples of first-order equations in this chapter. In later chapters we shall see that certain simple mechanical or electrical systems can be described by second-order equations such as

$$\frac{d^2u}{dt^2} + 6\frac{du}{dt} + 10u = 2\cos t.$$

More complex systems may require differential equations of yet higher order.

Our objective, wherever possible, is to solve differential equations. Let us symbolize a general first-order equation as

$$\frac{du}{dt} = F(t, u).$$

Then a *solution* of this differential equation on an interval $\alpha < t < \beta$ is a function $u(t)$ that has a first derivative and satisfies the differential equation for all t in the interval $\alpha < t < \beta$. That is, substitution of $u(t)$ into the differential equation leads to an identity,

$$\frac{d}{dt}u(t) = F(t, u(t)), \qquad \alpha < t < \beta.$$

Example 2

The differential equation of Example 1,

$$\frac{du}{dt} = -0.04u + 32,$$

has for one solution the function

$$u(t) = 800 + 70e^{-0.04t} \tag{1.3}$$

over the interval $-\infty < t < \infty$. To confirm this claim, first note that the given function has a first derivative, which is

$$\frac{d}{dt}u(t) = 70(-0.04)e^{-0.04t} = -2.8e^{-0.04t}.$$

Substitution of $u(t)$ and its derivative into the given differential equation leads to the identity

$$-2.8e^{-0.04t} = -0.04(800 + 70e^{-0.04t}) + 32, \qquad -\infty < t < \infty.$$

It is also correct to say that the more general expression

$$u(t) = 800 + ce^{-0.04t}, \tag{1.4}$$

in which c is an arbitrary constant, is a solution of the differential equation. Indeed, substitution of this function into the differential equation again gives

$$-0.04ce^{-0.04t} = -0.04(800 + ce^{-0.04t}) + 32,$$

which is true for all t and any choice of the constant c.

Returning now to the chemical reactor problem of Example 1, we seem to have an unexpected problem: too many answers. Since Eq. (1.4) is a solution of our differential equation for any value of c, and each different value of c corresponds to a different function, we have an infinite family of solutions of Eq. (1.2). Yet the physical problem seemed perfectly definite, and we expect a single, definite solution.

This difficulty disappears, however, when we note that there is information given in Example 1 that we have not used. The initial concentration in the tank was given to be 2 grams per liter, which translates to an initial amount of 400 grams. In terms of the function u, we would state this condition as

$$u(0) = 400. \tag{1.5}$$

Now if we set $t = 0$ in the function of Eq. (1.4), we get

$$u(0) = 800 + ce^0 = 800 + c.$$

This quantity should equal 400. Thus $c = -400$, and the function we seek is

$$u(t) = 800 - 400e^{-0.04t}. \tag{1.6}$$

This function satisfies both the differential equation (1.2) and the auxiliary condition (1.5).

Definition 1.2

A first-order differential equation, together with a condition on the value of the solution at some point (an *initial condition*) is called an *initial value problem.* A solution of the differential equation that also satisfies the initial condition is a solution of the initial value problem. A general first-order initial value problem is denoted by

$$\frac{du}{dt} = F(t, u), \qquad u(t_0) = q.$$

A substantial part of any course in calculus is actually spent in dealing with the problem of solving first-order differential equations in which the right-hand side is a known function of t alone:

$$\frac{du}{dt} = f(t). \tag{1.7}$$

In words: the derivative of an unknown function is given, and the function is to be found. A solution of this problem is any antiderivative or indefinite integral of $f(t)$. The theorems of elementary calculus assure us that the most general solution is obtained by adding a constant to any solution. If an initial condition is imposed, the constant can be chosen to make the solution satisfy it.

Example 3

Solve the initial value problem

$$\frac{du}{dt} = e^{-2t}, \qquad t > 0,$$

$$u(0) = 5.$$

The right-hand side of the differential equation is a known function of t. By "integrating both sides" of the differential equation we find

$$u(t) = -\frac{e^{-2t}}{2} + c$$

as a solution of the differential equation. In order to fulfill the initial condition we must have

$$u(0) = 5,$$

$$\frac{-e^0}{2} + c = -\frac{1}{2} + c = 5.$$

Thus $c = \frac{11}{2}$, and the solution of the initial value problem is

$$u(t) = \frac{11}{2} - \frac{e^{-2t}}{2}.$$

Some functions $f(t)$ do not have an antiderivative that can be written down in closed form. In this case we must leave the integration of $f(t)$ to be done. To make our solution of the differential equation

$$\frac{du}{dt} = f(t)$$

perfectly definite, we write it as

$$u(t) = c + \int_a^t f(z)\,dz. \tag{1.8}$$

The lower limit, a, is any convenient fixed value (usually the initial value of t in initial value problems). We have used z as the dummy variable of integration; any other letter that is not busy elsewhere could be used instead. Elementary theorems of calculus assure us that Eq. (1.8) is a continuous function whose derivative is $f(t)$ at any t where f is continuous. We also use the form (1.8) to represent the solution of the differential equation when we do not want to specify the function f. However, this should not be used as a "formula for solving" the differential equation (1.4). It is much easier and more natural to think of integrating both sides.

Example 4 We attempt to solve the initial value problem

$$\frac{du}{dt} = e^{-t^2}, \qquad u(0) = \tfrac{1}{2}.$$

There is no function expressible in terms of polynomials, exponentials, etc., whose derivative is e^{-t^2}; therefore we must leave the integration to be done. We express the solution of the differential equation as

$$u(t) = \int_0^t e^{-z^2}\,dz + c.$$

The initial condition can be satisfied by setting $t = 0$ in the expression above and equating $u(0)$ to $\frac{1}{2}$:

$$u(0) = \int_0^0 e^{-z^2}\,dz + c = \tfrac{1}{2}.$$

We see that $c = \frac{1}{2}$ and that the solution of the initial value problem is

$$u(t) = \int_0^t e^{-z^2}\,dz + \tfrac{1}{2}.$$

Exercises

In Exercises 1–10, you are to solve the given differential equation. If there is an initial condition, choose the constant of integration to satisfy it.

1. $\dfrac{du}{dt} = 4,\ 0 < t;\ u(0) = 1$
2. $\dfrac{du}{dt} = e^{-5t}$
3. $\dfrac{du}{dt} = \sin 2t,\ u(0) = 0$
4. $\dfrac{du}{dt} = \cos 3t$
5. $\dfrac{du}{dt} = \dfrac{1}{t+1},\ t > 0$
6. $\dfrac{du}{dt} = \dfrac{t}{1+t^2}$
7. $\dfrac{du}{dt} = \dfrac{t}{\sqrt{1+t^2}}$
8. $\dfrac{du}{dt} = \dfrac{1}{t(t+1)},\ t > 1$
9. $\dfrac{du}{dt} = \dfrac{t-1}{t^2+3t+2},\ t > 0$
10. $\dfrac{du}{dt} = 2t + 3$
11. A tank is being filled with water at a rate of $q(t)$ liters per minute. If the tank starts empty, find an initial value problem describing the volume of water in the tank. (Assume that no water leaves the tank.)
12. Solve the initial value problem of Exercise 11 if $q(t) = 8$ liters per minute. For how long is your solution valid if the tank has a capacity of 100 liters?
13. Solve the initial value problem of Exercise 11 if the flow rate in liters per minute is

$$q(t) = \begin{cases} 4 - \dfrac{t}{10}, & 0 < t \le 40 \text{ min} \\ 0, & 40 < t. \end{cases}$$

14. Solve the initial value problem of Exercise 11 if $q(t) = e^{-t/20}$, $0 < t$.
15. Find an initial value problem for the amount of salt in a 50-liter tank if pure water enters at a rate of 5 liters per minute, solution is drawn off at a rate of 5 liters per minute, and the tank is initially filled with brine at a concentration of 10 grams per liter.
16. Suppose that a tank has a capacity of V liters; brine enters at a rate of Q liters per minute and concentration k; the well-stirred contents of the tank are drawn off at a rate of Q liters per minute; the tank is initially filled with brine at a concentration k_0. Show that an initial value problem for the amount of salt in the tank, $u(t)$, is

$$\frac{du}{dt} = -\frac{Q}{V}u + Qk, \qquad 0 < t,$$

$$u(0) = Vk_0.$$

17. Suppose a solid object (like a salt block) is to be dissolved in a liquid. The rate at which the solid dissolves is $-dV/dt$, where V is the volume of the solid. It is reasonable to assume that this rate depends on the area A of the solid that is in contact with the liquid and on the difference $(c_s - c)$ between saturation concentration of the solution and its current concentration. In mathematical terms we have said (k is a constant of proportionality)

$$\frac{dV}{dt} = -kA(c_s - c).$$

(a) Suppose the solid is in the form of a sphere of radius R. Rephrase the equation above as a differential equation for R. (Recall $V = \frac{4}{3}\pi R^3$, $A = 4\pi R^2$.)
(b) Solve the equation obtained in (a), assuming that $c_s - c$ (approximately) constant. Designate $R(0) = R_0$.

18. Suppose now that the solid has a "characteristic dimension" L (the radius of a sphere or the side of a cube) and that its shape retains the same proportions as it shrinks. Then $V = vL^3$, $A = aL^2$, where v and a are constants. Derive a differential equation for L and solve it.

1.2 Linear Equations

In this section and the next we will study methods for solving a very important kind of first-order differential equation. A *linear* equation is one that can be expressed as

$$\frac{du}{dt} = a(t)u + f(t). \tag{1.9}$$

The key feature is that the unknown function appears in just one place, as the multiplier of a given function, $a(t)$. For the time being, we assume that both $a(t)$ and $f(t)$ are continuous over some interval $\alpha < t < \beta$ that we are interested in.

A linear equation is further classified as being *homogeneous* if $f(t)$ is identically 0. The general linear homogeneous equation of first order is thus

$$\frac{du}{dt} = a(t)u. \tag{1.10}$$

If $f(t)$ is not identically 0, the equation is *nonhomogeneous*, and $f(t)$ is the *inhomogeneity*. The constant function $u(t) \equiv 0$ is always a solution of a homogeneous linear equation and is never a solution of a nonhomogeneous one.

Example 1

In Section 1.1 we derived an equation to describe the amount of salt in a mixing tank:

$$\frac{du}{dt} = -0.04u + 32.$$

This equation is linear, since the right-hand side has the form prescribed by Eq. (1.9). We can identify $a(t) = -0.04$ and $f(t) = 32$, both constant functions. Since $f(t)$ is nonzero, the equation is nonhomogeneous.

A homogeneous linear first-order equation can be solved with one integration. The thought process of someone solving

$$\frac{du}{dt} = a(t)u$$

runs as follows:

First divide through the equation by u to obtain

$$\frac{1}{u}\frac{du}{dt} = a(t). \tag{1.11}$$

Now the left-hand side is the derivative of a familiar function, $\ln|u|$. If we integrate both sides, we get

$$\ln|u| = A(t) + C, \tag{1.12}$$

where $A(t)$ represents an indefinite integral of $a(t)$. To recover u itself, first exponentiate both sides:

$$|u| = e^{A(t)+C} = e^{A(t)}e^{C} \tag{1.13}$$