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To Sushiela and Emilia

Preface

This book contains a systematic exposition of the elements of the theory of dynamical systems in metric spaces with emphasis on the stability theory and its application and extension for ordinary autonomous differential equations.

In our opinion, the book should serve as a suitable text for courses and seminars in the theory of dynamical systems at the advanced undergraduate and beginning graduate level, in mathematics, physics and engineering.

It was never our intention to write a treatise containing all known results on the subject; but we have endeavored to include most of the important new results and developments of the past 20 years. The extensive bibliography at the end should enhance the usefulness of the book to those interested in the further exploration of the subject.

Students should have completed an elementary course in ordinary differential equations and have some knowledge of metric space theory, which is usually covered in undergraduate courses in analysis and topology.

Each author strongly feels that any mistakes left in the book are attributable to the other author, but each would appreciate receiving any comments from the scientific community.

We are obliged to Professor AARON STRAUSS for reading the entire typescript and pointing out several corrections. We would also like to thank Doctors FLORENCIO CASTILLO, LAWRENCE FRANKLIN, CESŁAW OLECH and GIULIO TRECCANI for help in proofreading the galleys.

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Notation

Set Theoretic Notation

Throughout the book standard set theoretic notations are used. Thus $\subset$, $\cup$, $\cap$, stand for set inclusion, set union and set intersections, respectively. For a given set M, ∂M, $\mathscr{I}M$, $\overline{M}$, $\mathscr{C}(M)$ denote the boundary, interior, closure, and complement of the set M, respectively. For given sets A, B, the set $A-B$ is the set difference.

Other standardly used set theoretic notations are:

X	a metric space with metric ϱ.
2^X	family of all subsets of X.
R	set of real numbers.
R^n	real n-dimensional euclidean space.
R^+	set of non-negative reals.
R^-	set of non-positive reals.
$\emptyset$	the empty set.
$\lvert\cdot\rvert$	absolute value of a real number.
$\lVert\cdot\rVert$	euclidean distance norm.
$\langle x, y\rangle$	the scalar product of vectors x, y in R^n.
$S(x, \alpha)$	for given $x \in X$ and $\varepsilon > 0$ is the open ball of radius $\alpha > 0$ centered at x, i.e., the set $\{y: \varrho(x, y) < \alpha\}$.
$S(M, \alpha)$	the set $\{y: \varrho(y, M) < \alpha\}$, where $M \subset X$ and $\alpha > 0$ are given.
$S[x, \alpha]$	the closed ball of radius $\alpha \geqq 0$ centered at x, i.e., the set $\{y: \varrho(x, y) \leqq \alpha\}$.
$S[M, \alpha]$	the set $\{y: \varrho(y, M) \leqq \alpha\}$.
$H(x, \alpha)$	the spherical hypersurface of radius $\alpha \geqq 0$ centered at x, i.e., the set $\{y: \varrho(x, y) = \alpha\}$.
$H(M, \alpha)$	the set $\{y: \varrho(y, M) = \alpha\}$.
$\{x_n\}$ or $\{x^n\}$	a sequence.
$x_n \to x$	sequence $\{x_n\}$ converges to x.
$\mathscr{C}^1$	family of continuously differentiable functions.
$\mathscr{C}^2$	family of twice continuously differentiable functions.

Notation Pertaining to Dynamical Systems

(X, R, π)	dynamical system on a space X (I, 1.1, p. 5).
π	phase map of a given dynamical system (I, 1.1, p. 5).
π^t	transition corresponding to a given $t \in R$ (p. 6).
π_x	motion through x (p. 6).
$\gamma(x)$	trajectory through x (II, 1.9, p. 14).
$\gamma^+(x)$ $(\gamma^-(x))$	positive (negative) semi-trajectory through x (II, 1.9, p. 14).

$\Lambda^+(x)$ $(\Lambda^-(x))$	positive (negative) limit set of x (II, 3.1, p. 19).
$D^+(x)$ $(D^-(x))$	positive (negative) prolongation of x (II, 4.1, p. 24).
$J^+(x)$ $(J^-(x))$	positive (negative) prolongational limit set of x (II, 4.1, p. 25).
$D^+_\alpha(x)$ $(D^-_\alpha(x))$	for given ordinal number α, the α-th positive (negative) prolongation of x (VII, 1.12, p. 123).
$J^+_\alpha(x)$ $(J^-_\alpha(x))$	for given ordinal number α, the α-th positive (negative) prolongational limit set of x (VII, 3.1, p. 129).
$D^+_u(M)$	uniform positive prolongation of a set M (VII, 2.11, p. 128).
$D^+(M, U)$	for given M, U in X, the positive prolongation of M, relative to U (II, 4.10, p. 29).
$A_\omega(M)$	region of weak attraction of $M \subset X$ (V, 1.1, p. 56).
$A(M)$	region of attraction of $M \subset X$ (V, 1.1, p. 56).
$A_u(M)$	region of uniform attraction of $M \subset X$ (V, 1.1, p. 56).
$\mathcal{R}$	the generalized recurrent set (VII, 3.6, p. 131).
$\mathcal{D}$	operator used in the definition of higher prolongation (VII, 1.1, p. 120).
$\mathcal{S}$	operator used in the definition of higher prolongation (VII, 1.1, p. 120).

Contents

Introduction

The theory of dynamical systems may be said to have begun as a special topic in the theory of ordinary differential equations with the pioneering work of HENRI POINCARÉ in the late 19th century. POINCARÉ, followed by IVAR BENDIXSON, studied topological properties of solutions of autonomous ordinary differential equations in the plane. The Poincaré-Bendixson theory is now a standard topic of discussion in courses on ordinary differential equations, and is adequately covered in all its details in the books of, say, CODDINGTON and LEVINSON [1], LEFSCHETZ [1], HARTMAN [1], SANSONE and CONTI [1], and NEMYTSKII and STEPANOV [1]. Of these, HARTMAN's book contains the most detailed and recent exposition.

Almost simultaneously with POINCARÉ, A.M. LIAPUNOV developed his theory of stability of a motion (solution) for a system of n first order ordinary differential equations. He defined in a precise form the concept of stability, asymptotic stability, and instability; and gave a "method" (the second or direct method of Liapunov) for the analysis of the stability properties of a given solution of an ordinary differential equation. Both his definition and his "method" characterize, in a strictly local setting, the stability properties of a solution of the differential equation. As such, the Liapunov theory is strikingly different from the Poincaré theory, in which, on the contrary, the study of the global properties of differential equations in the plane play a major role.

One of the main aspects of the Poincaré theory is the introduction of the concept of a trajectory, i.e., a curve in the x, $\dot{x}$ plane, parametrized by the time variable t, which can be found by eliminating the variable t from the given equations, thus reducing it to a first order differential equation connecting x and $\dot{x}$. In this way, POINCARÉ set up a convenient geometric framework in which to study qualitative behavior of planar differential equations. POINCARÉ was not interested in the integration of particular types of equations, but in classifying all possible behaviors of the class of all second order differential equations. By introducing this concept of trajectory, POINCARÉ was able to formulate and solve, as topological problems, problems in the theory of differential equations.

In the above fashion POINCARÉ paved the way for the formulation of the abstract notion of a dynamical system, which can be essentially attributed to A. A. MARKÓV and H. WHITNEY. These two authors separately noticed that one could study the qualitative theory of families of curves (trajectories) in a suitable space X, provided that these families are somehow restricted in their possible behavior, e.g., if they are defined, as having been generated by a general one-parameter topological transformation group acting on X.

Great impetus to the theory of dynamical systems came from the work of G. D. BIRKHOFF, who may truly be considered as the founder of the theory. His celebrated 1927 monograph on Dynamical Systems (BIRKHOFF [1]) is the basis of much of the research which came in the 1930's and 1940's and even today it is not outdated. BIRKHOFF established the two main streams of work on the theory of dynamical systems, namely, the topological theory and the ergodic theory.

In 1947, V. V. NEMYTSKII and V. V. STEPANOV [1] completed their "Qualitative Theory of Differential Equations" which to this day has served as a standard reference for all the major development in the theory of dynamical systems up to the middle 1940's. In 1949, NEMYTSKII [10] wrote a survey paper on the topological problems in the theory of dynamical systems, which sums up almost all the research into the topological theory to the end of 1940's.

During the 1950's a relatively large effort went into the generalization of the concept of a dynamical system to topological transformation groups. Thus in 1955 the book of W. H. GOTTSCHALK and G. A. HEDLUND [1] appeared, and a large body of research has appeared since in print. In this connection the work of R. ELLIS, H. FURSTENBERG, J. AUSLANDER, H. CHU, F. HAHN, S. KAKUTANI, besides GOTTSCHALK and HEDLUND, is noteworthy. On the other hand problems on structural stability in ordinary differential equations have led to the efforts at introducing the concepts and methods of differential topology and the theories of S. SMALE, D. V. ANOSOV, J. MOSER, M. PEIXOTO, and L. MARKUS. In this connection ANOSOV's monograph [1] is noteworthy.

More recently the basic theory has been extended by bringing in, the stability problems à la Liapunov, characteristically absent in earlier works on dynamical systems and topological transformation groups. In this connection, TARO URA's work, in particular his theory of prolongations, and its connections with stability has clearly shown that a significant portion of stability theory is topological in nature and hence belongs to the main stream of the theory of dynamical systems. An attempt in bringing in the direct method of Liapunov was made by V. I. Zubov [1]. However, Zubov mainly carried over to flows in metric spaces, results and methods previously known in differential equations without attempting to develop an independent theory.

The present volume was thus conceived to present in an easy and readable fashion the recent research in the theory of dynamical systems on metric spaces, and especially the stability theory together with its concrete applications to the theory of differential equations.

This book does not introduce several interesting areas of modern day research, such as the theory of structural stability (which requires some knowledge of differential topology), ergodic theory, and the general theory of topological transformation groups. To keep the presentation at a level easily accessible to undergraduate students in their junior or senior years (who have had some exposure to metric spaces and differential equations), we have not introduced local dynamical systems, although most results are true for such systems. Local semi-dynamical systems (N. P. Bhatia and O. Hajek [1]), flows without uniqueness (G. P. Szegö and G. Treccani [1]), are other major areas of development today. The present book should be helpful to all those desiring to work in any one of the above mentioned fields of study and research, which are not covered in this volume.

As to material covered in this volume, Chapters I—VII contain the basic theory of dynamical systems in metric spaces and Chapters VIII and IX contain applications and extensions of the stability theory (Chapter V) to dynamical systems defined by ordinary differential equations. Specifically, Chapter I contains the definition of a dynamical system and some examples to indicate various fields of application. Chapter II contains elementary notions which remain invariant under certain topological transformations of dynamical systems. Chapter III deals mainly with minimal sets and their structure. Chapter IV is devoted to the study of dispersive and parallelizable dynamical systems and concludes the part of the book devoted to the basic theory. Chapter V develops the main theme of the book, i.e., the stability and attraction theory. The theory presented here differs rather strongly from the one developed by Zubov, being essentially based on the concept of weak attraction (absent in Zubov's work). Chapter VI is devoted to a more specific problem: the classification of flows near a compact invariant set. Some results are given, but many problems in this are still open. Chapter VII contains the theory of higher prolongations originated by T. Ura with applications to absolute stability and generalized recurrence. Chapter VIII deals with the geometrical theory of stability for ordinary autonomous differential equations including various extensions of Liapunov's direct method. Chapter IX is again devoted to a more specific problem of characterizing stability and attraction concepts via non-continuous Liapunov functions; these are concepts like the weak attractor, which are not characterizable by continuous Liapunov functions.

Regarding the formal structure of the book, each chapter is divided into sections, followed by an un-numbered section of notes and references. In each section, individual items (Definitions, Theorems, etc.) are numbered consecutively. Each item may be subdivided into consecutively numbered subitems. References to the same chapter do not mention the chapter number. Thus, for example, reference 2.5.3 indicates, section 2, item 5, subitem 3 of the same chapter. References to other chapters contain an indication of the chapter number. Thus II, 3.17 denotes item 17 of section 3 in Chapter II. References to the bibliography are given by the author's name followed, if necessary, by an item number between brackets.

Chapter I

Dynamical Systems

In this chapter we introduce the definition of a dynamical system or what is also called a continuous flow. Several general examples are given to motivate and prepare the reader for the study of the theory of dynamical systems. Throughout the book the symbol X denotes a metric space with metric ϱ and R stands for the set of real numbers.

1. Definition and Related Notation

1.1 **Definition.** A *dynamical system* on X is the triplet (X, R, π), where π is a map from the product space $X \times R$ into the space X satisfying the following axioms:

1.1.1 $\pi(x, 0) = x$ for every $x \in X$ (identity axiom),

1.1.2 $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$ for every $x \in X$ and t_1, t_2 in R (group axiom),

1.1.3 π is continuous (continuity axiom).

Given a dynamical system on X, the space X and the map π are respectively called the *phase space* and the *phase map* (of the dynamical system). Unless otherwise stated a dynamical system on X is always assumed given.

In the sequel we shall generally delete the symbol π. Thus the image $\pi(x, t)$ of a point (x, t) in $X \times R$ will be written simply as xt. The identity and the group axioms then read

1.1.4 $x0 = x$ for every $x \in X$,

1.1.5 $xt_1(t_2) = x(t_1 + t_2)$ for every $x \in X$ and t_1, t_2 in R.

In line with this notation, if $M \subset X$ and $A \subset R$, then MA is the set $\{xt\colon x \in M \text{ and } t \in A\}$. If either M or A is a singleton, i.e., $M = \{x\}$ or

$A = \{t\}$, then we simply write xA and Mt for $\{x\}\,A$ and $M\{t\}$, respectively. For any $x \in X$, the set xR is called the trajectory through x (see II, 1.9).

The phase map determines two other maps when one of the variables x or t is fixed. Thus for fixed $t \in R$, the map $\pi^t\colon X \to X$ defined by $\pi^t(x) = xt$ is called a *transition*, and for a fixed $x \in X$, the map $\pi_x\colon R \to X$ defined by $\pi_x(t) = xt$ is called a *motion* (through x). Note that π_x maps R onto xR.

The following theorem expresses an important property of the transitions.

1.2 Theorem. For each $t \in R$, π^t is a homeomorphism of X onto itself.

Proof. For any $t \in R$ the transition π^t is continuous as π is such. To see that π^t is one-to-one and onto observe that if $yt = zt$, then $y = z$ follows from $y = y\,0 = y(t-t) = yt(-t) = zt(-t) = z(t-t) = z\,0 = z$. Again, if $y \in X$, then $\pi^t(x) = y$ for $x = y(-t)$ is easily verified. Finally to see that π^t has a continuous inverse we need only show that the transition π^{-t} is the inverse of π^t. To see this note that for any two transitions π^t and π^s, the composition $\pi^t \circ \pi^s$ is the transition π^{t+s}, because for any $x \in X$,

$$\pi^t \circ \pi^s(x) = \pi^t(\pi^s(x)) = \pi^t(xs) = xs(t) = x(t+s) = \pi^{t+s}(x).$$

Note also that the transition π^0 is the identity, since for any $x \in X$, $\pi^0(x) = x\,0 = x$. Since now $\pi^{-t} \circ \pi^t = \pi^{t-t} = \pi^0$, the transition π^{-t} is the inverse of π^t.

1.3 *Exercises.*

1.3.1 Show that the transitions π^t, $t \in R$, form a commutative group with the group operation being the composition of transitions.

1.3.2 For any $x \in X$ and $[a, b] \subset R$, the set $x[a, b]$ is compact and connected.

2. Examples of Dynamical Systems

2.1 *Ordinary Autonomous Differential Systems.* Consider the autonomous differential system

2.1.1 $$\frac{dx}{dt} = \dot{x} = f(x),$$

where $f\colon R^n \to R^n$ (R^n is the real n-dimensional euclidean space) is continuous and moreover assume that for each $x \in R^n$ a unique solution

$\varphi(t, x)$ exists which is defined on R and satisfies $\varphi(0, x) = x$. Then it is well known (see for example CODDINGTON and LEVINSON [1], chapters 1 and 2) that the uniqueness of solutions implies

2.1.2 $$\varphi(t_1, \varphi(t_2, x)) = \varphi(t_1 + t_2, x) \qquad \text{for } t_1, t_2 \text{ in } R$$

and considered as a function from $R \times R^n$ into R^n, φ is continuous in its arguments (section 4, chapter 2 in CODDINGTON and LEVINSON [1]). It is clear that the map $\pi: R^n \times R \to R^n$ such that $\pi(x, t) = \varphi(t, x)$ defines a dynamical system on R^n. We remark that the conditions on solutions of 2.1.1, as required above, are obtained, for example, if the function f satisfies a global Lipschitz condition, i.e., there is a positive number k such that

2.1.3 $$\|f(x) - f(y)\| \leqq k\,\|x - y\| \qquad \text{for all } x, y \text{ in } R^n.$$

2.2 *Ordinary Autonomous Differential Systems* (*Continued*). To illustrate that the theory of dynamical systems as defined in this chapter is applicable to a much larger class of ordinary autonomous differential systems we consider a system

2.2.1 $$\frac{dx}{dt} = \dot{x} = f(x), \qquad x \in R^n$$

where $f: D \to R^n$ is a continuous function on some open set $D \subset R^n$, and for each $x \in D$, 2.2.1 has a unique solution $\varphi(t, x)$, $\varphi(0, x) = x$ defined on a maximal interval (a_x, b_x), $-\infty \leqq a_x < 0 < b_x \leqq +\infty$. For each $x \in D$ define $\gamma^+(x) = \{\varphi(t, x)\colon\ 0 \leqq t < b_x\}$, and $\gamma^-(x) = \{\varphi(t, x)\colon a_x < t \leqq 0\}$. $\gamma^+(x)$ and $\gamma^-(x)$ are respectively called the positive and negative trajectory through the point $x \in D$. We will show that to each system 2.2.1, there corresponds a system

2.2.2 $$\frac{dx}{dt} = \dot{x} = g(x), \qquad x \in R^n,$$

where $g: D \to R^n$, such that 2.2.2 defines a dynamical system on D with the property that for each $x \in D$ the systems 2.2.1 and 2.2.2 have the same positive and the same negative trajectories. Thus in general it is sufficient to consider 2.2.2 instead of 2.2.1.

If $D = R^n$, then given 2.2.1, we set

2.2.3 $$\frac{dx}{dt} = \dot{x} = g(x) = \frac{f(x)}{1 + \|f(x)\|},$$

where $\|\cdot\|$ is the euclidean-distance norm. If $D \neq R^n$, then $\partial D \neq \emptyset$ and is closed. In this case, given 2.2.1, we set

2.2.4 $$\frac{dx}{dt} = \dot{x} = g(x) = \frac{f(x)}{1 + \|f(x)\|}\,\frac{\varrho(x, \partial D)}{1 + \varrho(x, \partial D)},$$