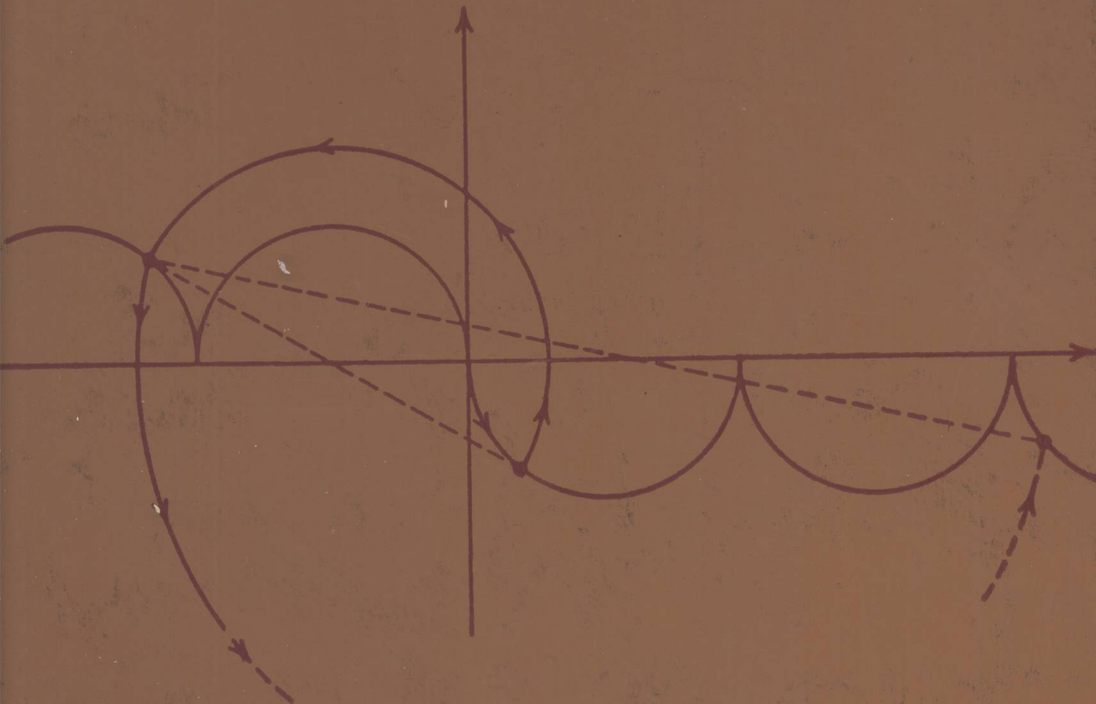
OSCILLATION THEORY OF OPTIMAL PROCESSES



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OSCILLATION THEORY OF OPTIMAL PROCESSES

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Some parts of the book translated by Robert H. Silverman

A Wiley-Interscience Publication

JOHN WILEY & SONS

New York • Chichester • Brisbane • Toronto • Singapore

Library of Congress Cataloging in Publication Data:

Smirnov, George M. (George Mikhael), 1945–
  Oscillation theory of optimal processes.

  "A Wiley-Interscience publication."
  Includes bibliographical references and indexes.
  1. Control theory. 2. Oscillations. I. Silverman, Robert H. II. Title.
QA402.3.S6285 1984 629.8'312 83-25941
ISBN 0-471-89399-4

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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The tom instantly jumped off the chair, and everyone saw that he had been sitting on a thick stack of manuscripts. He gave Woland the top one with a bow.

Mikhail Bulgakov,
"The Master and Margarita"

To my wife, *Alla*

PREFACE

This book is intended for use in graduate courses on the analysis and design of optimal dynamic systems and for independent study by engineers and applied mathematicians. The book presumes some knowledge of calculus, ordinary differential equations, and matrix algebra.

We will be studying the set of all control functions that satisfy the Pontryagin maximum principle. It is proved that, even in the most general case, this set, which also potentially contains the optimal control, can be substantially contracted. For this purpose, we prove a number of theorems that may be used to *split* the maximally feasible control set into nested classes according to *oscillation criteria.* An easily checked criterion makes it possible to simplify the computations substantially, even prior to computing the control. The linear theory of optimal processes developed by R. V. Gamkrelidze is shown to be a logical consequence of the oscillation theory presented here.

The monograph grew out of results that were first obtained in 1971–1977 and integrated as a theory in 1979. The present book includes additional material proved in 1980–1983, which essentially doubled the size of the original work. The book includes all the theorems necessary for the analysis, along with a large number of examples that illustrate, step by step, how to solve specific problems.

I wish to express my appreciation to Robert H. Silverman for his faithful translation of the original version of the manuscript (1980) and his careful editing of the expanded English version (1983). My greatest thanks go to my wife for the suggestions she made for improving the readability of the original version and for encouragement and support during the preparation of this book.

GEORGE M. SMIRNOV

Boston, Massachusetts
April 1984

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INTRODUCTION

Music, music before all things
Uneven rhythm suits it well
In air more vague and soluble
With nothing there that weighs or clings.

Paul Verlain, "The Art of Poetry"

In any area of human activity, we are constantly forced to select the best of a set of possible strategies, given particular conditions. In the present book we will be considering controllable processes that may be described by differential equations. The calculus of variations, a branch of mathematics with a history of over two hundred years, was the first mathematical tool used to solve optimal-control problems for such processes. However, even the earliest attempts to apply classical methods from the calculus of variations to contemporary practical problems faced major difficulties, since certain features of these problems could not be easily expressed in the classical formulation.

This fact led researchers to search for new techniques for solving practical problems. Two major results, both achieved in the 1950s, were Pontryagin's maximum principle and Bellman's dynamic-programming method. Together, they produced an interpretation of contemporary problems so distinct from that ordinarily encountered in the classical interpretation as to point to the emergence of a new branch of modern mathematics: optimal-control theory. This discipline now encompasses a rather large number of results.

Even though the development of optimal-control theory seems nearly complete, the actual process of solving particular problems often presents major difficulties. Though a number of more or less satisfactory solution algorithms for linear optimization problems are now available, there are only a few nonlinear problems that can be said to have been completely solved. Considerable difficulty may be created by special features concealed in certain practical problems (e.g, a solution that satisfies necessary optimality conditions may turn out to be nonunique, a designated class of feasible controls may lack an optimal control, and so on). Thus the development of special approaches is called for.

In the present book, a classification of controls is proposed. For each class of controls, we state conditions under which the optimal control belongs to it. The system of classes is practically complete, that is, practically any system trajectory may be placed in a particular class. Conversely, in each class the set of controls that satisfy the conditions of the maximum principle can be described as a p-parameter family of functions with p a pre-assigned finite or infinite number. The results yield general numerical procedures for solving boundary-value problems associated with the optimal-control problem for nonlinear systems. In many instances, only a few parameters are needed (p is small). Moreover, the phase space of the system can often be partitioned into domains, in each of which the number of parameters is constant. The initial controlled system can then be split into a series of systems, each with its own domain. This classification of controls is based on an analysis of the oscillatory properties of functions generated by the necessary optimality conditions of Pontryagin's maximum principle; hence our title.

The book consists of five chapters. The first chapter contains a description of the optimal-control problem we will be studying and a statement of the basic problem of *oscillation theory* presented in the book. The second chapter describes existing approaches to the analysis of *linear* systems based on concepts similar to oscillation theory. The third chapter presents *oscillation criteria* by means of which it is possible to prove theorems that supply a unique definition of an optimal control on the basis of the maximum principle, and also presents techniques for *singular-control* analysis. The fourth chapter extends the concept of an oscillation criterion to include *nonsingular* extremals with p corner points. In the fifth chapter may be found a presentation and evaluation of techniques for computing an optimal control using the analysis given in Chapters 3 and 4. Examples are provided throughout the book to illustrate the basic results. For the reader's convenience, we present below a list of these examples.

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Sec. 4.4

CHAPTER ONE

OSCILLATION CRITERIA IN THE THEORY OF OPTIMAL PROCESSES

mr youse needn't be so spry
concernin questions arty

each has his tastes but as for i
i likes a certain party

e. e. cummings

The present chapter may serve as an introduction to and link between the necessary optimality conditions of the maximum principle on the one hand, and the notion of *oscillation criteria* on the other. Everywhere below we will be considering the traditional statement of the optimal control problem and well-known results achieved with the maximum principle. The problem and its solution are outlined in the first part of this chapter and referred to throughout the book as the *optimal-control problem*. The second part of this chapter contains a general discussion of our central topic and states the *basic problem of oscillation theory*.

1.1. THE OPTIMAL-CONTROL PROBLEM AND THE MAXIMUM PRINCIPLE

Let us consider a controllable process that may be described by a system of differential vector equations of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_1) = \mathbf{x}_1. \tag{1}$$

Here $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is a vector that describes the state of the controllable process. The positions of the *controls* are determined by the control vector $\mathbf{u} = \{u_1, u_2, \ldots, u_r\}$. The vector function $\mathbf{f} = \{f^1, f^2, \ldots, f^n\}$ defines the rate of variation of the state vector $\mathbf{x}$.

Suppose that the scalar function

$$J = \int_{t_0}^{t_1} f^0(\mathbf{x}, \mathbf{u})\, dt, \tag{2}$$

defined by means of trajectories of the system (1) of differential equations, is to be minimized over the class $D_{\max}$ of measurable r-dimensional functions $\mathbf{u}(t)$, $t_0 \leq t \leq t_1$, with values taken from the set V. The moments t_0 and t_1 are considered fixed. [The control set V can be represented by a compact convex polyhedron (see Fig. 1) in many applications, though most of the theorems in this book do not specifically require it.]

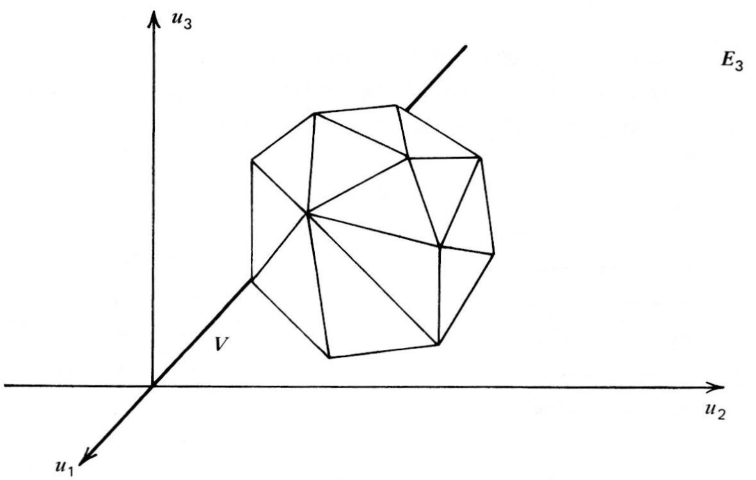

Fig. 1. Compact convex polyhedron V.

In the Pontryagin maximum principle [1], D_{max} is called a class of *feasible controls*. It is assumed here that the functions $f^0(\mathbf{x}, \mathbf{u}), f^1(\mathbf{x}, \mathbf{u}), \ldots, f^n(\mathbf{x}, \mathbf{u})$ are continuous together with their partial derivatives with respect to x.

The solution of the optimal control problem is provided by the control $\mathbf{u}(t)$ *(called the optimal control) and the trajectory* $\mathbf{x}(t)$ *of the system* (1) *corresponding to* $\mathbf{u}(t)$ *(called the optimal trajectory) that minimize the functional* (2).

The controllable process discussed above is given in automatic-control theory as shown in Fig. 2. The quantities $u_1, \ldots, u_r$ (control parameters) are frequently called *input variables*, and the quantities $x_1, \ldots, x_n$ (phase coordinates or state variables) are termed *output variables*. The output $\mathbf{J}$ is defined by the functional (2) and called *an optimality criterion*. The internal structure of the block in Fig. 2 is defined by the vector $\mathbf{f}$ and function f^0.

Example 1: Diesel Electric Propulsion Plant

Let us consider the power circuit of a diesel electric propulsion plant [2] described by a system of equations of the form

$$\frac{dx_1}{dt} = -a_1x_1 + k_2u - b_1x_2, \qquad\qquad (3)$$

$$\frac{dx_2}{dt} = a_2x_1 - b_2x_2 - b_3x_2^2,$$

where x_1 is the relative deviation of the speed of the diesel engine,
x_2 is the relative variation of the rate of rotation of the screw propeller, and
u is the relative deviation of the control rack. For the sake of simplicity, we let $|u| \leq 1$.

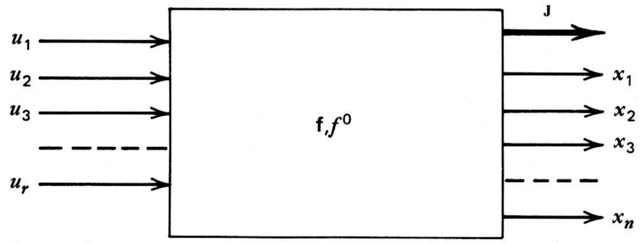

Fig. 2. Input-output object flow chart.

Suppose that we are interested in a control that provides minimal fuel consumption. In this case, the optimality criterion (2) is given as

$$J = \int_{t_0}^{t_1} u^2(t)\, dt \tag{4}$$

The optimal control $u(t)$ is the function that minimizes (4) and satisfies (3) when $t \in [t_0, t_1]$.

continued in Sec. 3.2

The existence of an optimal control for (1) and (2) has been discussed in a number of well-known studies [3–6]. However, it is not relevant to our principal theme, and so will be set aside. We will study only those controls that satisfy the necessary optimality conditions of the Pontryagin maximum principle [1]. Since the maximum principle is the necessary condition for optimality, the optimal control $u(t)$ belongs to the class of functions that satisfy the following condition:

For any optimal control $\mathbf{u}(t)$, *there exists a nonzero, continuous vector function*

$$\psi(t) = \{\psi_0(t), \psi_1(t), \ldots, \psi_n(t)\}$$

that satisfies, together with the optimal trajectory, a system of $2n + 2$ *differential equations*

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial \mathscr{H}}{\partial \psi_i}, \qquad i = 0, 1, 2, \ldots, n, \\ \frac{d\psi_i}{dt} &= -\frac{\partial \mathscr{H}}{\partial x_i}, \qquad i = 0, 1, 2, \ldots, n, \end{aligned} \tag{5}$$

such that the function

$$\mathscr{H} = \mathscr{H}(\psi(t), \mathbf{x}(t), \mathbf{u}) = \psi(t) \cdot \mathbf{f}(\mathbf{x}(t), \mathbf{u}) = \sum_{\alpha=0}^{n} \psi_\alpha(t) f^\alpha(\mathbf{x}(t), \mathbf{u})$$

of the variable $\mathbf{u} \in V$ *attains almost everywhere on the segment* $[t_0, t_1]$ *its maximal value*

$$\mathscr{H}(\psi(t), \mathbf{x}(t), \mathbf{u}(t)) \,(=)\, M(\psi(t), \mathbf{x}(t)) \tag{6}$$

at the point $\mathbf{u} = \mathbf{u}(t)$ *(the sign* $(=)$ *denotes equal almost everywhere), where*

$$M(\psi(t), \mathbf{x}(t)) = \sup_{\mathbf{u} \in V} \mathscr{H}(\psi(t), \mathbf{x}(t), \mathbf{u}). \tag{7}$$

In this case the following inequalities hold at the moment of time t_1:

$$\psi_0(t_1) \le 0, \qquad M(\mathbf{x}(t_1), \boldsymbol{\psi}(t_1)) = 0. \tag{8}$$

Further, if the functions $\boldsymbol{\psi}(t)$, $\mathbf{x}(t)$, *and* $\mathbf{u}(t)$ *satisfy the system* (5) *and condition* (6), *the functions* $\psi_0(t)$ *and* $M(\boldsymbol{\psi}(t), \mathbf{x}(t))$ *of the variable* t *are constant, so that the inequalities of* (8) *can be verified for any moment* t, $t_0 \le t \le t_1$, *and not just for the single moment* t_1.

Bear in mind that the functional (2) does not occur in the statement of the maximum principle. Instead, there is a differential equation in the system (5) that corresponds to the equation

$$\frac{dx_0}{dt} = f^0(\mathbf{x}, \mathbf{u}), \qquad x_0(t_0) = 0, \quad x_0(t_1) = J. \tag{9}$$

This is why the system (5) is of dimension $2n + 2$, and not $2n$, as might be expected. In the theory of differential equations, systems such as (5) are said to be *canonical*, while the function $\mathscr{H}$ is called a *Hamiltonian*.

The process of finding an optimal control is usually interpreted as that of mapping a control function $\mathbf{u}(t)$ whose graph remains inside the control polyhedron V (Fig. 3) onto a trajectory $\mathbf{x}(t) = \{x_0(t), \ldots, x_n(t)\}$ in the phase space, where the component $x_0(t)$ is minimal.

The feasible control $\mathbf{u}^*$, $t \in [t_0, t_1]$, is said to be Pontryagin-extremal if it satisfies, together with the corresponding functions $\mathbf{x}^*(t)$ and $\boldsymbol{\psi}^*(t)$, the maximum condition (6). Restating the maximum principle, we may say that every optimal control is a *Pontryagin extremal.* The optimal-control problem therefore reduces to finding the control in a set of Pontryagin extremals. If

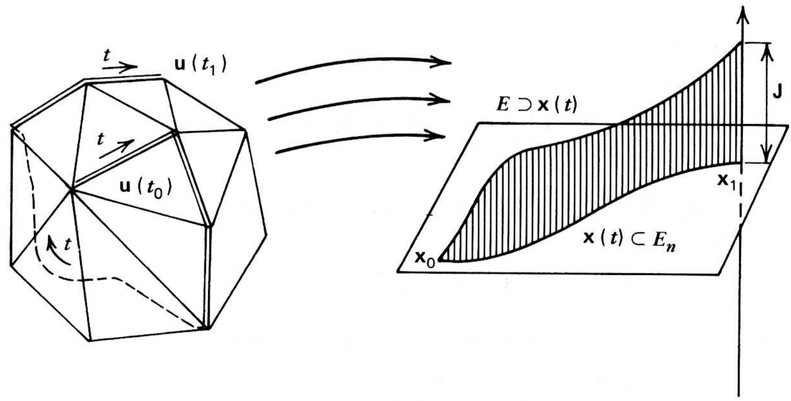

Fig. 3. Mapping a control function $\mathbf{u}(t)$ onto the trajectory $\mathbf{x}(t)$.

an optimal control exists in some optimization problem, and if the Pontryagin extremal is unique, it is also the optimal control.

Example 2: Reservoir with Integrating Power Unit at the Inlet

The object is described by the system of equations [7] (see Fig. 4)

$$\frac{dx_1}{dt} = u,$$
$$\frac{dx_2}{dt} = \left(x_1 - \sqrt{x_2}\right)\frac{1}{S}, \tag{10}$$

where x_1 is the fluid consumed,
x_2 is the fluid level, and
S is the design constant of the outlet.

The boundary conditions are $t_0 = 0$, $x_{10} = x_{20} = 0$, $t_1 = t_{\text{opt}}$, $x_1 = x_{11}$, and $x_2 = x_{21}$. The Hamiltonian and canonical system of equations (5) are as follows:

$$\mathscr{H} = \psi_0 \cdot 1 + \psi_1 u + \psi_2 \cdot \left(x_1 - \sqrt{x_2}\right)\frac{1}{S}$$

$$\dot{x}_0 = 1, \qquad \dot{x}_1 = u, \qquad \dot{x}_2 = \left(x_1 - \sqrt{x_2}\right)\frac{1}{S} \tag{11}$$

$$\dot{\psi}_0 = 0, \qquad \dot{\psi}_1 = -\frac{\psi_2}{S}, \qquad \dot{\psi}_2 = \frac{\psi_2}{2S\sqrt{x_2}}.$$

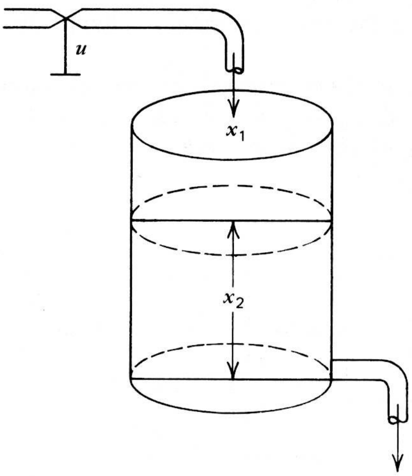

Fig. 4. Reservoir with integrating power unit at the inlet.