VOLUME ONE

CALCULUS

AND ANALYTIC GEOMETRY

FOBES AND SMYTH

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AND ANALYTIC GEOMETRY

MELCHER P. FOBES

Professor and Chairman
Department of Mathematics
The College of Wooster

RUTH B. SMYTH

Associate Professor
Department of Mathematics
The College of Wooster

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to our teachers:

EDWARD S. HAMMOND and CHESTER H. YEATON

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PREFACE

This volume is the first of two planned to serve as a two-year course in calculus and analytic geometry and to prepare the student to continue to a standard, rigorous course in advanced calculus. The present volume deals with the basic techniques and applications of differentiation and integration, while its companion extends these topics and deals with series, elements of differential equations, and an introduction to the differentiation and integration of functions of several arguments.

We have written this book in the hope that it may help the students who read it thoughtfully to obtain a real understanding of the basic structure, concepts, and techniques of the early parts of the calculus, and that it may foster their ability to draw from these principles the methods of attacking and solving those problems whose key lies in the calculus. Our own students, at any rate, come to their first course in calculus for a variety of reasons, ranging from desire for a background for the physical sciences to a straight love affair with mathematics and the abstract, and they come with a wide range of maturity in their understanding of algebra and with different degrees of facility in technique. All these considerations have governed our plan to write them a four-semester text whose beginning they could find understandable at their original level of mathematical maturity and whose degree of rigor and sophistication would grow with their ability to appreciate it.

We presuppose for the text a fairly fluent recall of algebra through quadratics (which we test in the freshman orientation period) and a semester's worth of analytic trigonometry which has frittered away a minimum of time measuring flagpoles and has concentrated on the fundamental identities and the solution of trigonometric equations.

The book is long and wordy — intentionally so. There are several first-rate, concise, elegant texts on the calculus. For the sake of our own students — and we hope for others also — we have tried to be more detailed, more conversational. Particularly at the beginning we have first introduced ideas intuitively, with scant care for rigor, but with an earnest effort to make more straightforward and simple the transition from elementary courses to the calculus. This transition is hard enough at best, moving as it does from courses where many problems can be done after one pattern to one in which clear and full understanding of principles is indispensable if a single mind is to handle all the welter of problems whose solutions these principles make possible. We have found, for instance, that the first chapter, with its intuitive, informal approach to the notion of limit, has been most helpful in bridging the gap between high school algebra and this new concept, and in avoiding the common misunderstandings which seem to accompany early encounters with the derivative as a limit.

But, though an idea may first be met informally and without probing, it recurs throughout the text, each time more searchingly examined; for we have tried to bring concepts bit by bit to a fully rigorous statement, so far as time and the maturity of the student will allow. As the text proceeds and the student becomes more of a mathematician, less preamble is essential, but we still make an effort to throw as many lights as possible on the growing structure. The student who finishes the book should be easily able to cope with the rigors of a sound course in advanced calculus.

The book has not been written for a class composed only of brilliant students; it aims at a group of all abilities, so long as they have true mastery of the prerequisites we have mentioned. We have held to the viewpoint that a course of formulas and problems, however rapidly it may precipitate the student into advanced physics, runs counter to the main purposes of seeing patterns of thought and understanding principles which make possible the applications. Thus we have not spared the theory, and we expect our students to know it — and be examined on it. (Accordingly, for example, we proceed slowly with the early chapters on analytic geometry where the proofs are simple and the instructor can make a point of careful demonstration.) And this applies to the weaker as well as the stronger students. These latter may find explanations here which are more detailed than they need; let them turn to the sections on "Things to Think About" at the ends of the chapters, which offer stronger meat.

These "Things to Think About" provide matter of supplementary interest and sometimes of greater difficulty or subtlety, which the good student may find interesting to mull over. Some are simple; some more complex. Especially in the latter portions of the book, these sections develop additional portions of theory which are not in the mainstream of the course or which serve to link the present text to advanced calculus. Witness, for example, the development of the theorem on integrability of a continuous function in the "Things to Think About" for Chapter 13, or the definition of the logarithmic function by an integral in Chapter 17. While they are not intended as part of the formal course, nor for the use of weaker students, they do furnish new and often significant insights for the proficient.

How much about limits and continuity to bring in has, of course, been a problem. ϵ- and δ-methods are rather carefully discussed, not from the very first page of the book — this seems to us unprofitably early — but beginning after the student has had a chance to get his bearings in a new land, and continuing with increasing emphasis throughout. Theorems on limits and continuity are stated carefully when they are used, and in Volume II sample ϵ- and δ-proofs of limit theorems are given as a supplement to a chapter on series, but there is no systematic development of these topics. It has seemed to us that this should all be brought together formally later, when the student who continues mathematical training can grasp it in half the time.

Our especial thanks go to Professor Albert A. Bennett for a careful reading of the manuscript and for a good many helpful suggestions for its betterment, and to our recent colleagues in the department who have given the book the benefit of their wisdom and experience in teaching it. We must also thank our students who braved for several years the mimeographed form of the book, and who, perhaps stimulated by a reward of $1 for each mathematical mistake they found, have been startlingly assiduous word-by-word readers. They have saved us quite a few dollars' worth of boners; and, naturally, they are to blame for any that they have been careless enough to leave undetected!

M. P. F.

R. B. S.

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1

INTRODUCTION: INTUITIVE APPROACH TO LIMITS

1.1 THE CALCULUS, ANALYTIC GEOMETRY, AND YOU

I. If a bullet is fired with a certain muzzle velocity into the air at an angle, how far will it go? When will it be highest in the air? How high will it be then? How fast is it traveling at any particular time?

II. If at a given time a satellite is traveling about the earth in a certain direction and with a certain velocity, can we predict the path along which it will travel thereafter?

III. If the supporting cable of a suspension bridge hangs from two supports, in what kind of curve will it hang? How high will it be above the level of the bridge at any particular point?

IV. If an observer has listening posts in safe territory, how does he determine the position of an enemy gun by observations of its sound made at those posts?

V. Can a map of the earth be made on a piece of paper so that areas, distances, directions, and scales may be completely preserved? If not, which ones can be preserved on any given map? How can such a map be constructed accurately?

VI. What is the smallest amount of tin necessary to make a cylindrical one-quart can? What dimensions will the can have?

VII. The total cost of manufacturing a certain article is made up of: (1) fixed costs of $1000 a day; (2) production cost of 22¢ per article; (3) cost of repairs, etc., which is $x^2/5500$ dollars per day, where x is the number of articles produced per day. How many articles should be produced each day to make the cost per unit the least?

VIII. If you wish to design a reflector for a headlight so that every ray from the light will be reflected parallel to a given direction, how should you shape the reflector?

IX. If the ends of a steel rail are fixed, but the rail increases in length because of an increase in temperature, what is the maximum amount by which the rail will buckle from its straight position if it buckles into an arc of a parabola?

X. Where is the center of gravity of a hemisphere?

XI. How are the values in a table of logarithms found?

XII. If a gas expands in a cylinder, how much work is done by the expanding gas?

XIII. How much oil is there in a given cylindrical tank which lies on its side, if the depth of the oil is one-fourth of the diameter?

Here are some of the problems—a very random sample of them, by the way, and far from representative of the whole variety—which can be faced and answered with the help of the calculus and analytic geometry. We shall find the answers to some of them in this course; others require more technique than this book can cover. Most of them will have to be phrased more precisely before they are stated with enough accuracy to be answered.

Questions similar to these go back as far as Archimedes (third century B.C.) or even further. The real answers did not come, however, until the 1600's, when Descartes took the decisive step which established analytic geometry, and somewhat later Isaac Newton and Gottfried Leibniz, practically simultaneously though independently, hit upon the ideas which brought to fulfilment the attempts of their predecessors and created the branches of mathematics known as the differential and the integral calculus.

Most of the ideas involved in these subjects were not new. They still dealt with numbers, geometric figures, and the techniques of algebra and geometry. The chief major addition was a process called "taking the limit," with which we shall have much to do in the course of our work. The insight of these men perceived methods of tying algebra, geometry, and the new process together in ways that had not been thought of before. As with most