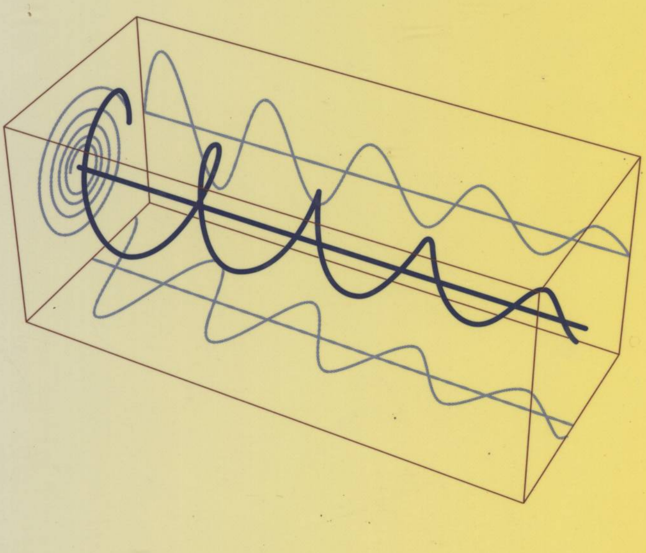
Clay C. Ross

DIFFERENTIAL EQUATIONS

An Introduction with Mathematica®

Second Edition



Springer

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With 88 Figures

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(continued after index)

This book is dedicated to my parents,
Vera K. and Clay C. Ross,
and to my wife,
Andrea,
with special gratitude to each.

Preface

Goals and Emphasis of the Book

Mathematicians have begun to find productive ways to incorporate computing power into the mathematics curriculum. There is no attempt here to use computing to avoid doing differential equations and linear algebra. The goal is to make some first explorations in the subject accessible to students who have had one year of calculus. Some of the sciences are now using the symbol-manipulative power of *Mathematica* to make more of their subject accessible. This book is one way of doing so for differential equations and linear algebra.

I believe that if a student's first exposure to a subject is pleasant and exciting, then that student will seek out ways to continue the study of the subject. The theory of differential equations and of linear algebra permeates the discussion. Every topic is supported by a statement of the theory. But the primary thrust here is obtaining solutions and information about solutions, rather than proving theorems. There are other courses where proving theorems is central. The goals of this text are to establish a solid understanding of the notion of solution, and an appreciation for the confidence that the theory gives during a search for solutions. Later the student can have the same confidence while personally developing the theory.

When a study of the book has been completed, many important elementary concepts of differential equations and linear algebra will have been encountered. In addition, the use of *Mathematica* makes it possible to analyze problems that are formidable without computational assistance. *Mathematica* is an integral part of the presentation, because in introductory differential equations or linear algebra courses it is too often true that simple tasks like finding an antiderivative, or finding the roots of a polynomial of relatively high degree—even when the roots are all rational—completely obscure the mathematics that is being studied. The complications encountered in the manual solution of a realistic problem of four first-order linear equations with constant coefficients can totally obscure the beauty and centrality of the theory. But having *Mathematica* available to carry out the complicated steps frees the student to think about what is happening, how the ideas work together, and what everything means.

The text contains many examples. Most are followed immediately by the same example done in *Mathematica*. The form of a *Mathematica* notebook is reproduced almost exactly so that the student knows what to expect when trying problems by him/herself. Having solutions by *Mathematica* included in the text also provides a sort of encyclopedia of working approaches to doing things in *Mathematica*. In addition, each of these examples exists as a real *Mathematica* notebook that can be executed, studied, printed out, or modified to do some other problem. Other *Mathematica* notebooks may be provided by the instructor. Occasionally a problem will request that new methods be tried, but by the time these occur, students should be able to write effective *Mathematica* code of their own.

Mathematica can carry the bulk of the computational burden, but this does not relieve the student of knowing whether or not what is being done is correct. For that reason, periodic checking of results is stressed. Often an independent manual calculation will keep a *Mathematica* calculation safely on course. *Mathematica*, itself, can and should do much of the checking, because as the problems get more complex, the calculations get more and more complicated. A calculation that is internally consistent stands a good chance of being correct when the concepts that are guiding the process are correct.

Since all of the problems except those that are of a theoretical nature can be solved and checked in *Mathematica*, very few of the exercises have answers supplied. As the student solves the problems in each section, they should save the notebooks to disk—where they can serve as an answer book and study guide if the solutions have been properly checked. A *Mathematica* package is a collection of functions that are designed to perform certain operations. Several notebooks depend heavily on a package that has been provided. Most of the packages supplied undertake very complicated tasks, where the functions are genuinely intimidating, so the code does not appear in the text of study notebooks.

What Is New in This Edition

The changes are two-fold:

1. Rearrange and restate some topics (Linear algebra has now been gathered into a separate chapter, and series methods for systems have been eliminated.) Many typographical errors have been corrected.
2. Completely rewrite, and occasionally expand, the *Mathematica* code using version 5 of *Mathematica*.

In addition, since *Mathematica* now includes a complete and fully on-line Help subsystem, several appendices have been eliminated.

Topics Receiving Lesser Emphasis

The solutions of most differential equations are not simple, so the solutions of such equations are often examined numerically. We indicate some ways to have *Mathematica* solve differential equations numerically. Also, properties of a solution are often deduced from careful examination of the differential equation itself, but an extended study of qualitative differential equations must wait for a more advanced course. The best advice is to use the `NDSolve` function when a numerical solution is required.

Some differential equations have solutions that are very hard to describe either analytically or numerically because the equations are sensitive to small changes in the initial values. Chaotic behavior is a topic of great current interest; we present some examples of such equations, but do not fully develop the concepts.

Acknowledgments

I would like to thank those students and others who read the manuscript for the first edition, and several science department colleagues for enduring questions and for responding so kindly.

Reviews of the first edition were received from Professors Matthew Richey, Margie Hale, Stephen L. Clark, Stan Wagon, and William Sit. Dave Withoff contributed to that edition his expert help on technical aspects of *Mathematica* programming. Any errors that remain in this edition are solely the responsibility of the author.

Sewanee, Tennessee, USA
November 2003

CLAY C. ROSS

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About Differential Equations

1.0 Introduction

What Are Differential Equations? Who Uses Them?

The subject of differential equations is large, diverse, powerful, useful, and full of surprises. Differential equations can be studied on their own—just because they are intrinsically interesting. Or, they may be studied by a physicist, engineer, biologist, economist, physician, or political scientist because they can model (quantitatively explain) many physical or abstract systems. Just what is a differential equation? A differential equation having y as the dependent variable (unknown function) and x as the independent variable has the form

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0$$

for some positive integer n. (If n is 0, the equation is an algebraic or transcendental equation, rather than a differential equation.) Here is the same idea in words:

Definition 1.1. *A* **differential equation** *is an equation that relates in a nontrivial manner an unknown function and one or more of the derivatives or differentials of that unknown function with respect to one or more independent variables.*

The phrase "in a nontrivial manner" is added because some equations that appear to satisfy the above definition are really identities. That is, they are always true, no matter what the unknown function might be. An example of such an equation is:

$$\sin^2\left(\frac{dy}{dx}\right) + \cos^2\left(\frac{dy}{dx}\right) = 1.$$

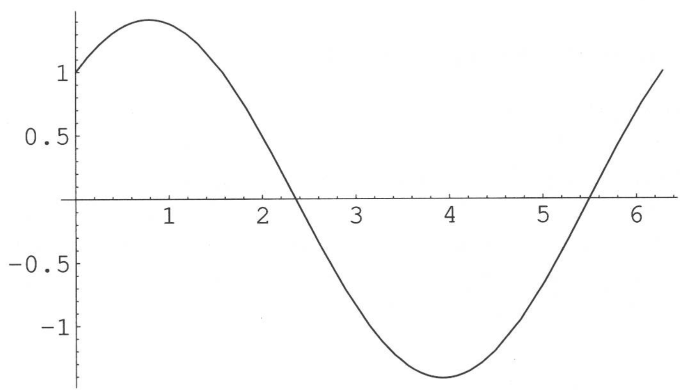
This equation is satisfied by every differentiable function of one variable. Another example is:

$$\left(\frac{dy}{dx} - y\right)^2 = \left(\frac{dy}{dx}\right)^2 - 2y\left(\frac{dy}{dx}\right) + y^2.$$

This is clearly just the binomial squaring rule in disguise: $(a + b)^2 = a^2 + 2ab + b^2$; it, too, is satisfied by every differentiable function of one variable. We want to avoid calling such identities differential equations.

One quick test to see that an equation is not merely an identity is to substitute some function such as $\sin(x)$ or e^x into the equation. If the result is ever false, then the equation is not an identity and is perhaps worthy of our study. For example, substitute $y = \sin(x)$ into $y' + y = 0$. The result is $\cos(x) + \sin(x) = 0$, and this is not identically true. (It is false when $x = \pi$, for instance.) If you have a complicated function and are unsure whether or not it is identically 0, you can use *Mathematica* to plot the function to see if it ever departs from 0. This does not constitute a proof, but it is evidence, and it suggests where to look if the function is not identically 0. A plot can be produced this way:

```
In[1]:= Plot[Cos[x] + Sin[x], {x, 0, 2π}];
```



Note that `Pi` is the symbol π in disguise. The π symbol can be found in the `BasicInput` palette.

Another extreme that we would like to avoid is an equation that is never true for real functions, such as

$$\left(\frac{dy}{dx}\right)^2 + y^2 = -1$$

No matter what the real differentiable function y is, the left-hand side of the equation is nonnegative and the right-hand side is negative—and this cannot happen. So the equations we want to study are those that can have some solutions, but not too many solutions. The meaning of this will become clear as we proceed. Unless stated otherwise, the solutions we seek will be real.

Classification of Differential Equations

Differential equations are classified in several different ways: **ordinary** or **partial**; **linear** or **nonlinear**. There are even special subclassifications: **homogeneous** or

nonhomogeneous; **autonomous** or **nonautonomous**; **first-order**, **second-order**, ... , **n th-order**. Most of these names for the various types have been inherited from other areas of mathematics, so there is some ambiguity in the meanings. But the context of any discussion will make clear what a given name means in that context. There are reasons for these classifications, the primary one being to enable discussions about differential equations to focus on the subject matter in a clear and unambiguous manner. Our attention will be on ordinary differential equations. Some will be linear, some nonlinear. Some will be first-order, some second-order, and some of higher order than second. What is the **order** of a differential equation?

Definition 1.2. *The* **order** *of a differential equation is the order of the highest derivative that appears (nontrivially) in the equation.*

At this early stage in our studies, we need only be able to distinguish ordinary from partial differential equations. This is easy: a differential equation is an **ordinary differential equation** if the only derivatives of the unknown function(s) are ordinary derivatives, and a differential equation is a **partial differential equation** if the only derivatives of the unknown function(s) are partial derivatives.

Example 1.1 Here are some ordinary differential equations:

$$\frac{dy}{dt} = 1 + y^2 \qquad \text{(first-order)} \qquad \text{[nonlinear]}$$

$$\frac{d^2y}{dx^2} + y = 3\cos(x) \qquad \text{(second-order)} \qquad \text{[linear, nonhomogeneous]}$$

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 5y = 0 \qquad \text{(third-order)} \qquad \text{[linear, homogeneous]}$$

Example 1.2 Here are some partial differential equations:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \qquad \text{(first-order in } x \text{ and } y\text{)}$$

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{(first-order in } t\text{; second-order in } x\text{)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(second-order in } x \text{ and } y\text{)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 3 \qquad \text{(second-order)}$$

Solutions of Differential Equations

Definition 1.3. *To say that* $y = g(x)$ *is a* **solution** *of the differential equation*

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{dy}{dx^n}\right) = 0$$

on an interval I means that

$$F(x, g(x), g'(x), \ldots, g^{(n)}(x)) = 0$$

for every choice of x in the interval I. In other words, a solution, when substituted into the differential equation, makes the equation identically true for x in I.

Example 1.3 The function $y = e^{-x}$ is a solution of the differential equation $y' + y = 0$, because $y' + y = -e^{-x} + e^{-x} = 0$ for all x. ◇

To have *Mathematica* verify this for you, conduct this dialog in an active *Mathematica* window:

```
In[2]:= Clear[x, y, a]

In[3]:= y[x_] = Exp[-x]
Out[3]= e^-x

In[4]:= y'[x] + y[x] == 0
Out[4]= True
```

The `True` that *Mathematica* returned indicates that $y'(x) + y(x) = 0$ (always), and hence we indeed have a solution. It is not necessary to `Clear` variables regularly, but if you get some unusual behavior, `Clear` the names involved, re-define them, and try the calculation again. *Mathematica* remembers definitions you may have forgotten, and these may interfere with a subsequent calculation.

Here are other examples of solutions of ordinary differential equations. They are from the notebook *Solutions of DE's*. You should execute ideas such as these yourself in *Mathematica*.

```
In[5]:= y[x_] = c Exp[x^2]
Out[5]= c e^(x^2)

In[6]:= Simplify[y'[x] - 2x y[x] == 0]
Out[6]= True

In[7]:= Clear[y]

In[8]:= y[t_] = c1 Sin[a t] + c2 Cos[a t]
Out[8]= c2 Cos[a t] + c1 Sin[a t]

In[9]:= Simplify[y''[t] + a^2 y[t] == 0]
Out[9]= True
```

Direction Fields and Solutions

The solutions of the first-order differential equation $dy/dx = f(x, y)$ can be represented nicely by a picture. Given a point $P = (x, y)$, the differential equation tells what the slope of the tangent line to a solution is at the point P. If m is such a slope then the differential equation says that

$$m = \left.\frac{dy}{dx}\right|_P = f(P) = f(x, y).$$

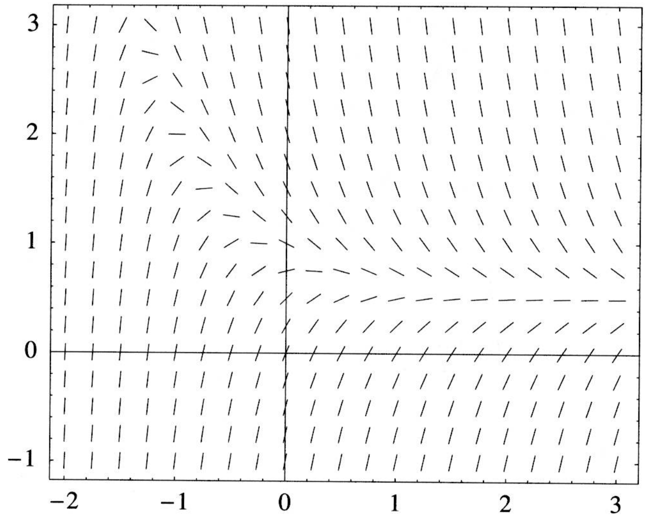


Fig. 1.1. A portion of the direction field of $dy/dx = (3/2) - 3y + e^{-3x/2}$.

The idea of a direction field is similar to that of a vector field, where $f(x, y)$, instead of giving a vector that is to be associated with (x, y), gives a slope that is to be associated with (x, y). If representatives of these slopes are indicated on a graph at enough points, some visual indication of the behavior of the solutions of the differential equation is suggested.

For example, in Figure 1.1 we have plotted some representative members of the direction field associated with the differential equation $dy/dx = (3/2) - 3y + e^{-3x/2}$. Then in Figure 1.2 some solutions of the differential equation are superimposed on the direction field. Notice how the direction field gives a sense of the behavior of the solutions. Solutions may be close together, but they do not cross. You may use the notebook *Direction Field Example* to produce similar pictures. These can help you understand the behavior of the solutions of any differential equation that has the form $dy/dx = f(x, y)$.

How Many Solutions Are There?

Once we understand that some differential equations have solutions, it is natural to ask several questions. How many solutions can a given differential equation have? (In general there are many; they may be easy or extremely difficult to find.) When there are many solutions to choose from, is it possible to select one or more having certain properties? When, if ever, is there exactly one solution having the properties we want?