Graduate Texts in Mathematics

Haakan Hedenmalm
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Theory of Bergman Spaces

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Preface

Their memorials are covered by sand,
their rooms are forgotten.
But their names live on by the books they wrote,
for they are beautiful.

(Egyptian poem, 1500–1000 BC)

The theory of Bergman spaces experienced three main phases of development during the last three decades.

The early 1970's marked the beginning of function theoretic studies in these spaces. Substantial progress was made by Horowitz and Korenblum, among others, in the areas of zero sets, cyclic vectors, and invariant subspaces. An influential presentation of the situation up to the mid 1970's was Shields' survey paper "*Weighted shift operators and analytic function theory*".

The 1980's saw the thriving of operator theoretic studies related to Bergman spaces. The contributors in this period are numerous; their achievements were presented in Zhu's 1990 book "*Operator Theory in Function Spaces*".

The research on Bergman spaces in the 1990's resulted in several breakthroughs, both function theoretic and operator theoretic. The most notable results in this period include Seip's geometric characterization of sequences of interpolation and sampling, Hedenmalm's discovery of the contractive zero divisors, the relationship between Bergman-inner functions and the biharmonic Green function found by

Duren, Khavinson, Shapiro, and Sundberg, and deep results concerning invariant subspaces by Aleman, Borichev, Hedenmalm, Richter, Shimorin, and Sundberg.

Our purpose is to present the latest developments, mostly achieved in the 1990's, in book form. In particular, graduate students and new researchers in the field will have access to the theory from an almost self-contained and readable source.

Given that much of the theory developed in the book is fresh, the reader is advised that some of the material covered by the book has not yet assumed a final form.

The prerequisites for the book are elementary real, complex, and functional analysis. We also assume the reader is somewhat familiar with the theory of Hardy spaces, as can be found in Duren's book "*Theory of H^p Spaces*", Garnett's book "*Bounded Analytic Functions*", or Koosis' book "*Introduction to H^p Spaces*".

Exercises are provided at the end of each chapter. Some of these problems are elementary and can be used as homework assignments for graduate students. But many of them are nontrivial and should be considered supplemental to the main text; in this case, we have tried to locate a reference for the reader.

We thank Alexandru Aleman, Alexander Borichev, Bernard Pinchuk, Kristian Seip, and Sergei Shimorin for their help during the preparation of the book. We also thank Anders Dahlner for assistance with the computer generation of three pictures, and Sergei Treil for assistance with one.

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Contents

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The Bergman Spaces

In this chapter we introduce the Bergman spaces and concentrate on the general aspects of these spaces. Most results are concerned with the Banach (or metric) space structure of Bergman spaces. Almost all results are related to the Bergman kernel. The Bloch space appears as the image of the bounded functions under the Bergman projection, but it also plays the role of the dual space of the Bergman spaces for small exponents ($0 < p \leq 1$).

1.1 Bergman Spaces

Throughout the book we let $\mathbb{C}$ be the complex plane, let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

be the open unit disk in $\mathbb{C}$, and let

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

be the unit circle in $\mathbb{C}$. Likewise, we write $\mathbb{R}$ for the real line. The normalized area measure on $\mathbb{D}$ will be denoted by dA. In terms of real (rectangular and polar) coordinates, we have

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta, \qquad z = x + iy = re^{i\theta}.$$

We shall freely use the Wirtinger differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$

where again $z = x + iy$. The first acts as differentiation on analytic functions, and the second has a similar action on antianalytic functions.

The word *positive* will appear frequently throughout the book. That a function f is positive means that $f(x) \geq 0$ for all values of x, and that a measure μ is positive means that $\mu(E) \geq 0$ for all measurable sets E. When we need to express the property that $f(x) > 0$ for all x, we say that f is *strictly positive*. These conventions apply – *mutatis mutandis* – to the word *negative* as well. Analogously, we prefer to speak of increasing and decreasing functions in the less strict sense, so that constant functions are both increasing and decreasing.

We use the symbol $\sim$ to indicate that two quantities have the same behavior asymptotically. Thus, $A \sim B$ means that A/B is bounded from above and below by two positive constants in the limit process in question.

For $0 < p < +\infty$ and $-1 < \alpha < +\infty$, the *(weighted) Bergman space* $A^p_\alpha = A^p_\alpha(\mathbb{D})$ of the disk is the space of analytic functions in $L^p(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z).$$

If f is in $L^p(\mathbb{D}, dA_\alpha)$, we write

$$\|f\|_{p,\alpha} = \left[\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z)\right]^{1/p}.$$

When $1 \leq p < +\infty$, the space $L^p(\mathbb{D}, dA_\alpha)$ is a Banach space with the above norm; when $0 < p < 1$, the space $L^p(\mathbb{D}, dA_\alpha)$ is a complete metric space with the metric defined by

$$d(f, g) = \|f - g\|^p_{p,\alpha}.$$

Since $d(f, g) = d(f - g, 0)$, the metric is invariant. The metric is also p-homogeneous, that is, $d(\lambda f, 0) = |\lambda|^p d(f, 0)$ for scalars $\lambda \in \mathbb{C}$. Spaces of this type are called *quasi-Banach spaces*, because they share many properties of the Banach spaces.

We let $L^\infty(\mathbb{D})$ denote the space of (essentially) bounded functions on $\mathbb{D}$. For $f \in L^\infty(\mathbb{D})$ we define

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)| : z \in \mathbb{D}\}.$$

The space $L^\infty(\mathbb{D})$ is a Banach space with the above norm. As usual, we let H^∞ denote the space of bounded analytic functions in $\mathbb{D}$. It is clear that H^∞ is closed in $L^\infty(\mathbb{D})$ and hence is a Banach space itself.

PROPOSITION 1.1 *Suppose $0 < p < +\infty$, $-1 < \alpha < +\infty$, and that K is a compact subset of $\mathbb{D}$. Then there exists a positive constant $C = C(n, K, p, \alpha)$ such that*

$$\sup\left\{|f^{(n)}(z)| : z \in K\right\} \leq C \, \|f\|_{p,\alpha}$$

for all $f \in A^p_\alpha$ and all $n = 0, 1, 2, \ldots$. In particular, every point-evaluation in $\mathbb{D}$ is a bounded linear functional on A^p_α.

Proof. Without loss of generality we may assume that

$$K = \{z \in \mathbb{C} : |z| \le r\}$$

for some $r \in (0, 1)$. We first prove the result for $n = 0$.

Let $\sigma = (1 - r)/2$ and let $B(z, \sigma)$ denote the Euclidean disk at z with radius σ. Then by the subharmonicity of $|f|^p$,

$$|f(z)|^p \le \frac{1}{\sigma^2} \int_{B(z,\sigma)} |f(w)|^p \, dA(w)$$

for all $z \in K$. It is easy to see that for all $z \in K$ we have

$$1 - |z|^2 \ge 1 - |z| \ge (1 - r)/2.$$

Thus, we can find a positive constant C (depending only on r) such that

$$|f(z)|^p \le C \int_{B(z,\sigma)} |f(w)|^p \, dA_\alpha(w) \le C \int_{\mathbb{D}} |f(w)|^p \, dA_\alpha(w)$$

for all $z \in K$. This proves the result for $n = 0$.

By the special case we just proved, there exists a constant $M > 0$ such that $|f(\zeta)| \le M \|f\|_{p,\alpha}$ for all $|\zeta| = R$, where $R = (1 + r)/2$. Now if $z \in K$, then by Cauchy's integral formula,

$$f^{(n)}(z) = \frac{n\,!}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}}.$$

It follows that

$$|f^{(n)}(z)| \le \frac{n\,! M R}{\sigma^{n+1}} \|f\|_{p,\alpha}$$

for all $z \in K$ and $f \in A^p_\alpha$. ■

As a consequence of the above proposition, we show that the Bergman space A^p_α is a Banach space when $1 \le p < +\infty$ and a complete metric space when $0 < p < 1$.

PROPOSITION 1.2 *For every $0 < p < +\infty$ and $-1 < \alpha < +\infty$, the weighted Bergman space A^p_α is closed in $L^p(\mathbb{D}, dA_\alpha)$.*

Proof. Let $\{f_n\}_n$ be a sequence in A^p_α and assume $f_n \to f$ in $L^p(\mathbb{D}, dA_\alpha)$. In particular, $\{f_n\}_n$ is a Cauchy sequence in $L^p(\mathbb{D}, dA_\alpha)$. Applying the previous proposition, we see that $\{f_n\}_n$ converges uniformly on every compact subset of $\mathbb{D}$. Combining this with the assumption that $f_n \to f$ in $L^p(\mathbb{D}, dA_\alpha)$, we conclude that $f_n(z) \to f(z)$ uniformly on every compact subset of $\mathbb{D}$. Therefore, f is analytic in $\mathbb{D}$ and belongs to A^p_α. ■

In many applications, we need to approximate a general function in the Bergman space A^p_α by a sequence of "nice" functions. The following result gives two commonly used ways of doing this.

PROPOSITION 1.3 *For an analytic function f in $\mathbb{D}$ and $0 < r < 1$, let f_r be the dilated function defined by $f_r(z) = f(rz)$, $z \in \mathbb{D}$. Then*

(1) For every $f \in A^p_\alpha$, we have $\|f_r - f\|_{p,\alpha} \to 0$ as $r \to 1^-$.

(2) For every $f \in A^p_\alpha$, there exists a sequence $\{p_n\}_n$ of polynomials such that $\|p_n - f\|_{p,\alpha} \to 0$ as $n \to +\infty$.

Proof. Let f be a function in A^p_α. To prove the first assertion, let δ be a number in the interval $(0, 1)$ and note that

$$\begin{aligned}\int_{\mathbb{D}} |f_r(z) - f(z)|^p \, dA_\alpha(z) &\leq \int_{|z|\leq\delta} |f_r(z) - f(z)|^p \, dA_\alpha(z) \\ &+ \int_{\delta<|z|<1} \left(|f_r(z)| + |f(z)|\right)^p dA_\alpha(z).\end{aligned}$$

Since f is in $L^p(\mathbb{D}, dA_\alpha)$, we can make the second integral above arbitrarily small by choosing δ close enough to 1. Once δ is fixed, the first integral above clearly approaches 0 as $r \to 1^-$.

To prove the second assertion, we first approximate f by f_r and then approximate f_r by its Taylor polynomials. ■

Although any function in A^p_α can be approximated (in norm) by a sequence of polynomials, it is not always true that a function in A^p_α can be approximated (in norm) by its Taylor polynomials. Actually, such approximation is possible if and only if $1 < p < +\infty$; see Exercise 4.

We now turn our attention to the special case $p = 2$. By Proposition 1.2 the Bergman space A^2_α is a Hilbert space. For any nonnegative integer n, let

$$e_n(z) = \sqrt{\frac{\Gamma(n + 2 + \alpha)}{n!\,\Gamma(2 + \alpha)}}\, z^n, \qquad z \in \mathbb{D}.$$

Here, $\Gamma(s)$ stands for the usual Gamma function, which is an analytic function of s in the whole complex plane, except for simple poles at the points $\{0, -1, -2, \dots\}$. It is easy to check that $\{e_n\}_n$ is an orthonormal set in A^2_α. Since the set of polynomials is dense in A^2_α, we conclude that $\{e_n\}_n$ defined above is an orthonormal basis for A^2_α. It follows that if

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{+\infty} b_n z^n$$

are two functions in A^2_α, then

$$\|f\|_2^2 = \sum_{n=0}^{+\infty} \frac{n!\,\Gamma(2 + \alpha)}{\Gamma(n + 2 + \alpha)} |a_n|^2$$

and

$$\langle f, g\rangle_\alpha = \sum_{n=0}^{+\infty} \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} a_n \overline{b}_n,$$

where $\langle \cdot, \cdot \rangle_\alpha$ is the inner product in A^2_α inherited from $L^2(\mathbb{D}, dA_\alpha)$.

PROPOSITION 1.4 *For $-1 < \alpha < +\infty$, let $\mathbf{P}_\alpha$ be the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α. Then*

$$\mathbf{P}_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)\,dA_\alpha(w)}{(1 - z\overline{w})^{2+\alpha}}, \qquad z \in \mathbb{D},$$

for all $f \in L^2(\mathbb{D}, dA_\alpha)$.

Proof. Let $\{e_n\}_n$ be the orthonormal basis of A^2_α defined a little earlier. Then for every $f \in L^2(\mathbb{D}, dA_\alpha)$ we have

$$\mathbf{P}_\alpha f = \sum_{n=0}^{+\infty} \langle \mathbf{P}_\alpha f, e_n\rangle_\alpha \, e_n.$$

In particular,

$$\mathbf{P}_\alpha f(z) = \sum_{n=0}^{+\infty} \langle \mathbf{P}_\alpha f, e_n\rangle_\alpha \, e_n(z)$$

for every $z \in \mathbb{D}$ and the series converges uniformly on every compact subset of $\mathbb{D}$. Since

$$\langle \mathbf{P}_\alpha f, e_n\rangle_\alpha = \langle f, \mathbf{P}_\alpha e_n\rangle_\alpha = \langle f, e_n\rangle_\alpha,$$

we have

$$\begin{aligned}
\mathbf{P}_\alpha f(z) &= \sum_{n=0}^{+\infty} \frac{\Gamma(n+2+\alpha)}{n!\,\Gamma(2+\alpha)} \int_{\mathbb{D}} f(w)(z\overline{w})^n \, dA_\alpha(w) \\
&= \int_{\mathbb{D}} f(w) \left[\sum_{n=0}^{+\infty} \frac{\Gamma(n+2+\alpha)}{n!\,\Gamma(2+\alpha)} (z\overline{w})^n \right] dA_\alpha(w) \\
&= \int_{\mathbb{D}} \frac{f(w)\,dA_\alpha(w)}{(1 - z\overline{w})^{2+\alpha}}.
\end{aligned}$$

The interchange of integration and summation is justified, because for each fixed $z \in \mathbb{D}$, the series

$$\sum_{n=0}^{+\infty} \frac{\Gamma(n+2+\alpha)}{n!\,\Gamma(2+\alpha)} (z\overline{w})^n$$

converges uniformly in $w \in \mathbb{D}$. ∎

The operators $\mathbf{P}_\alpha$ above are called the *(weighted) Bergman projections* on $\mathbb{D}$. The functions

$$K_\alpha(z,w) = \frac{1}{(1-z\overline{w})^{2+\alpha}}, \qquad z, w \in \mathbb{D},$$

are called the (weighted) Bergman kernels of $\mathbb{D}$. These kernel functions play an essential role in the theory of Bergman spaces.

Although the Bergman projection $\mathbf{P}_\alpha$ is originally defined on $L^2(\mathbb{D}, dA_\alpha)$, the integral formula

$$\mathbf{P}_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)\, dA_\alpha(w)}{(1-z\overline{w})^{2+\alpha}}$$

clearly extends the domain of $\mathbf{P}_\alpha$ to $L^1(\mathbb{D}, dA_\alpha)$. In particular, we can apply $\mathbf{P}_\alpha$ to a function in $L^p(\mathbb{D}, dA_\alpha)$ whenever $1 \le p < +\infty$.

If f is a function in A^2_α, then $\mathbf{P}_\alpha f = f$, so that

$$f(z) = \int_{\mathbb{D}} \frac{f(w)\, dA_\alpha(w)}{(1-z\overline{w})^{2+\alpha}}, \qquad z \in \mathbb{D}.$$

Since this is a pointwise formula and A^2_α is dense in A^1_α, we obtain the following.

COROLLARY 1.5 *If f is a function in A^1_α, then*

$$f(z) = \int_{\mathbb{D}} \frac{f(w)\, dA_\alpha(w)}{(1-z\overline{w})^{2+\alpha}}, \qquad z \in \mathbb{D},$$

and the integral converges uniformly for z in every compact subset of $\mathbb{D}$.

This corollary will be referred to as the reproducing formula. The Bergman kernels are special types of reproducing kernels.

On several occasions later on theorems will hold only for the unweighted Bergman spaces. Thus, we set $A^p = A^p_0$ and call them the ordinary Bergman spaces. The corresponding Bergman projection will be denoted by $\mathbf{P}$, and the Bergman kernel in this case will be written as

$$K(z,w) = \frac{1}{(1-z\overline{w})^2}.$$

The Bergman kernel functions are intimately related to the Möbius group Aut $(\mathbb{D})$ of the disk. To see this, let $z \in \mathbb{D}$ and consider the Möbius map φ_z of the disk that interchanges z and 0,

$$\varphi_z(w) = \frac{z-w}{1-\overline{z}w}, \qquad w \in \mathbb{D}.$$

We list below some basic properties of φ_z, which can all be checked easily.

PROPOSITION 1.6 *The Möbius map φ_z has the following properties:*

(1) $\varphi_z^{-1} = \varphi_z$.

(2) The real Jacobian determinant of φ_z *at* w *is* $|\varphi_z'(w)|^2 = \dfrac{(1-|z|^2)^2}{|1-z\overline{w}|^4}$.

(3) $1-|\varphi_z(w)|^2 = \dfrac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}$.

As a simple application of the properties above, we mention that the formula for the Bergman kernel function $K_\alpha(z,w)$ can be derived from a simple change of variables, instead of using an infinite series involving the Gamma function. More specifically, if $f \in A^1_\alpha$, then the rotation invariance of dA_α gives

$$f(0) = \int_{\mathbb{D}} f(w)\, dA_\alpha(w).$$

Replacing f by $f \circ \varphi_z$, making an obvious change of variables, and applying properties (2) and (3) above, we obtain

$$f(z) = (1-|z|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{f(w)\, dA_\alpha(w)}{(1-w\overline{z})^{2+\alpha}(1-z\overline{w})^{2+\alpha}}.$$

Fix $z \in \mathbb{D}$, and replace f by the function $w \mapsto (1-w\overline{z})^{2+\alpha} f(w)$. We then arrive at the reproducing formula

$$f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+\alpha}}\, dA_\alpha(w), \qquad z \in \mathbb{D},$$

for $f \in A^1_\alpha$. From this we easily deduce the integral formula for the Bergman projection $\mathbf{P}_\alpha$.

1.2 Some L^p Estimates

Many operator-theoretic problems in the analysis of Bergman spaces involve estimating integral operators whose kernel is a power of the Bergman kernel. In this section, we present several estimates for integral operators that have proved very useful in the past. In particular, we will establish the boundedness of the Bergman projection $\mathbf{P}_\alpha$ on certain L^p spaces.

THEOREM 1.7 *For any* $-1 < \alpha < +\infty$ *and any real* β, *let*

$$I_{\alpha,\beta}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-z\overline{w}|^{2+\alpha+\beta}}\, dA(w), \qquad z \in \mathbb{D},$$

and

$$J_\beta(z) = \int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^{1+\beta}}, \qquad z \in \mathbb{D}.$$