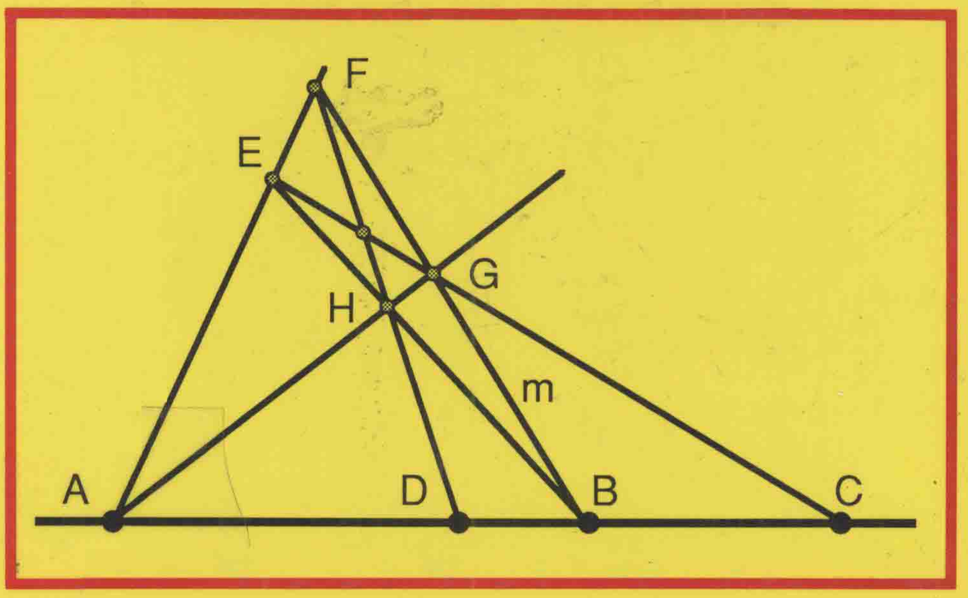
Undergraduate Texts in Mathematics

Judith N. Cederberg

A Course in Modern Geometries



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To Jim,
Anna, and Rachel

Preface

The origins of geometry are lost in the mists of ancient history, but geometry was already the preeminent area of Greek mathematics over 20 centuries ago. As such, it became the primary subject of Euclid's *Elements*. *Elements* was the first major example of a formal axiomatic system and became a model for mathematical reasoning. However, the eventual discoveries of non-Euclidean geometries profoundly affected both mathematical and philosophical understanding of the nature of mathematics. The relation between Euclidean and non-Euclidean geometries became apparent with the development of projective geometry—a geometry with origins in artists' questions about perspective.

This interesting historical background and the major philosophical questions raised by developments in geometry are virtually unknown to current students, who often view geometry as a dead subject full of two-column proofs of patently clear results. It is no surprise that Mary Kantowski, in an article entitled "Impact of Computing on Geometry," has called geometry "the most troubled and controversial topic in school mathematics today" (Fey, 1984, p. 31). However, this and many other recent articles provide evidence for an increasing realization that the concepts and methods of geometry are becoming more important than ever in this age of computer graphics. The geometry of the artists, projective geometry, has become the tool of computer scientists and engineers as they work on the frontiers of CAD/CAM (computer-aided design/computer-aided manufacturing) technology.

The major emphasis of this text is on the geometries developed after Euclid's *Elements* (circa 300 B.C.). In addition to the primary goal of studying these "newer" geometries, this study provides an excellent opportunity to explore aspects of the history of mathematics. Also, since algebraic techniques are frequently used, this study demonstrates the interaction of several areas of mathematics and serves to develop geometrical insights into mathematical results that previously appeared to be completely abstract in nature.

Since Euclid's geometry is historically the first major example of an axiomatic system and since one of the major goals of teaching geometry in high school is to expose students to deductive reasoning, Chapter 1 begins with a general description of axiomatic (or deductive) systems. Following this general introduction, several finite geometries are presented as examples of specific systems. These finite geometries not only demonstrate some of the concepts that occur in the geometries of Chapters 2 through 4 but also indicate the breadth of geometrical study.

In Chapter 2, Euclid's geometry is first covered in order to provide historical and mathematical preparation for the major topic of non-Euclidean geometries. This brief exposure to Euclid's system serves both to recall familiar results of Euclidean geometry and to show how few substantial changes have occurred in Euclidean geometry since Euclid formulated it. The non-Euclidean geometries are then introduced to demonstrate that these geometries, which appear similar to Euclidean geometry, have properties that are radically different from comparable Euclidean properties.

The beginning of Chapter 3 serves as a transition from the synthetic approach of the previous chapters to the analytic treatment contained in the remainder of this chapter and the next. There follows a presentation of Klein's definition of geometry, which emphasizes geometrical transformations. The subsequent study of the transformations of the Euclidean plane begins with isometries and similarities and progresses to the more general transformations called affinities.

By using an axiomatic approach and generalizing the transformations of the Euclidean plane, Chapter 4 offers an introduction to projective geometry and demonstrates that this geometry provides a general framework within which the geometries of Chapters 2 and 3 can be placed.

Although the text ends here, mathematically the next logical step in this process is the study of topology, which is usually covered in a separate course.

This text is designed for college-level survey courses in geometry. Many of the students in these courses are planning to pursue secondary-school teaching. However, with the renewed interest in geometry, other students interested in further work in mathematics or computer science will find the background provided by these courses increasingly valuable. These survey courses can also serve as an excellent vehicle for demonstrating the relationships between mathematics and other liberal arts disciplines. In an attempt to encourage student reading that further explores these relationships, each chapter begins with a section that lists suggested bibliographic sources for relevant topics in art, history, applications, and so on. I have found that having groups of students research and report on these topics not only introduces them to the wealth of expository writing in mathematics but also provides a way to share their acquired insights into the liberal arts nature of mathematics.

The material contained in this text is most appropriate for junior or senior mathematics majors. The only geometric prerequisite is some familiarity with the most elementary high-school geometry. Since the text makes frequent use of matrix algebra and occasional references to more general concepts of linear algebra, a background in elementary linear algebra is helpful. Because the text introduces the concept of a group and explores properties of geometric transformations, a course based on this text provides excellent preparation for the standard undergraduate course in abstract algebra.

I am especially grateful for the patient support of my husband and the general encouragement of my colleagues in the St. Olaf Mathematics Department. In particular, I wish to thank our department chair, Theodore Vessey, for his support and our secretary, Donna Brakke, for her assistance. I am indebted to the many St. Olaf alumni of Math 80 who studied from early drafts of the text and to Charles M. Lindsay for his encouragement after using preliminary versions of the text in his courses at Coe College in Cedar Rapids, Iowa. Others who used a preliminary version of the text and made helpful suggestions are Thomas Q. Sibley of St. John's University in Collegeville, Minnesota, and Martha L. Wallace of St. Olaf College. I am also indebted to Joseph Malkevitch of York College of the City University of New York for serving as mathematical reader for the text, and to Christina Mikulak for her careful editorial work.

Judith N. Cederberg

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CHAPTER 1

Axiomatic Systems and Finite Geometries

1.1. Gaining Perspective

Finite geometries were developed in the late 19th century, in part to demonstrate and test the axiomatic properties of "completeness," "consistency," and "independence." They are introduced in this chapter to fulfill this historical role and to develop both an appreciation for and an understanding of the revolution in mathematical and philosophical thought brought about by the development of non-Euclidean geometry. In addition, finite geometries provide relatively simple axiomatic systems in which we can begin to develop the skills and techniques of geometric reasoning. The finite geometries introduced in Sections 1.3 and 1.5 also illustrate some of the fundamental properties of non-Euclidean and projective geometry.

Even though finite geometries were developed as abstract systems, mathematicians have applied these abstract ideas in designing statistical experiments using Latin squares and in developing error-correcting codes in computer science. Section 1.4 develops a simple error-correcting code and shows its connection with finite projective geometries. The application of finite affine geometries to the building of Latin squares is equally intriguing. Since Latin squares are clearly described in several readily accessible sources, the reader is encouraged to explore this topic by consulting the resources listed at the end of this chapter.

1.2. Axiomatic Systems

The study of any mathematics requires an understanding of the nature of deductive reasoning, and geometry has been singled out for introducing this methodology to secondary-school students. There are important historical reasons for choosing geometry to fulfill this role, but these reasons are seldom revealed to secondary-school initiates. This section introduces the terminology essential for a discussion of deductive reasoning so that the extraordinary influence of the history of geometry on the modern understanding of deductive systems will become evident.

Deductive reasoning takes place in the context of an organized logical structure called an *axiomatic* (or *deductive*) system. Such a system consists of the following components:

1. Undefined terms.
2. Defined terms.
3. Axioms.
4. A system of logic.
5. Theorems.

Undefined terms are included since it is not possible to define all terms without resorting to circular definitions. In geometrical systems these undefined terms frequently, but not necessarily, include "point," "line," "plane," and "on." Defined terms are not actually necessary, but in nearly every axiomatic system certain phrases involving undefined terms are used repeatedly. Thus it is more efficient to substitute a new term, that is, a defined term, for each of these phrases whenever they occur. For example, in Euclidean geometry we substitute the term "parallel lines" for the phrase "lines which do not intersect." Furthermore, it is impossible to prove all statements constructed from the defined and undefined terms of the system without circular reasoning, just as it is impossible to define all terms. So an initial set of statements is accepted without proof. The statements that are accepted without proof are known as *axioms.* From the axioms, other statements can be deduced or proved using the rules of inference of a system of logic (usually Aristotelian). These latter statements are called *theorems.*

As noted earlier, the axioms of a system must be statements constructed using the terms of the system. But they cannot be arbitrarily constructed since an axiom system must be consistent.

Definition 1.1. An axiomatic system is said to be *consistent* if there do not exist in the system any two axioms, any axiom and theorem, or any two theorems that contradict each other.

It should be clear that it is essential that an axiomatic system be consistent since a system in which both a statement and its negation can be proved is worthless. However, it soon becomes evident that it would be difficult to verify consistency directly from this definition since all possible theorems would have to be considered. Instead, models are used for establishing consistency. A *model* of an axiomatic system is obtained by assigning interpretations to the undefined terms so as to convert the axioms into true statements in the interpretations. If the model is obtained by using interpretations that are objects and relations adapted from the real world, we say we have established *absolute consistency.* In this case, statements corresponding to any contradictory theorems would lead to contradictory statements in the model, but contradictions in the real world are supposedly impossible. On the other hand, if the interpretations assigned are taken from another axiomatic system, we have only tested consistency relative to the consistency of the second axiomatic system; that is, the system we are testing is consistent only if the system within which the interpretations are assigned is consistent. In this second case, we say we have demonstrated *relative consistency* of the first axiomatic system. Because of the number of elements in many axiomatic systems, relative consistency is the best we are able to obtain. We illustrate the use of models to determine consistency of the axiomatic system for four-point geometry.

Axioms for Four-Point Geometry

Undefined Terms. Point, line, on.
Axiom 1. There exist exactly four points.
Axiom 2. Two distinct points are on exactly one line.
Axiom 3. Each line is on exactly two points.

Before demonstrating the consistency of this system, it may be helpful to make some observations about these three statements, which will also apply to other axioms in this text. Axiom 1 explicitly guarantees the existence of exactly four points. However, even though lines are mentioned in Axioms 2 and 3, we cannot ascertain whether or not lines exist until theorems verifying this are proved since there is no axiom that explicitly insures their existence. This is true even though in this system the proof of the existence of lines is almost immediate. Axioms 2 and 3 like many mathematical statements are disguised "if...then" statements. Axiom 2 should be interpreted as follows: If two distinct points exist, then these two points are on exactly one line. Similarly, Axiom 3 should be interpreted: If there is a line, it is on exactly two points. In other axiomatic systems, we will discover that the axioms actually lead to theorems telling us that there are many more points and/or lines than those guaranteed to exist by the axioms.

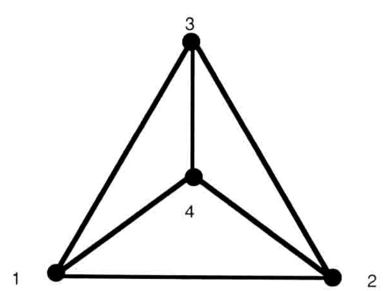

Figure 1.1

These observations suggest that the construction of any model for four-point geometry must begin with the objects known to exist, that is, four points. In model 1 these points are interpreted as the letters A, B, C, D whereas in model 2 (see Fig. 1.1) these points are interpreted as dots. In continuing to build either model, we must interpret the remaining undefined terms so as to create a system in which Axioms 2 and 3 become true statements.

Model 1

Undefined Term	Interpretation
Points	Letters A, B, C, D
Lines	Columns of letters given below
On	Contains or is contained in

$$\begin{matrix} A & A & A & B & B & C \\ B & C & D & C & D & D \end{matrix}$$

Model 2

Undefined Term	Interpretation
Points	Dots denoted 1, 2, 3, 4
Lines	Segments illustrated in Fig. 1.1
On	A dot is an endpoint of a segment or vice versa

There are several other important properties that an axiomatic system *may* possess.

Definition 1.2. An axiom in an axiomatic system is *independent* if it cannot be proved from the other axioms. If each axiom of a system is independent, the system is said to be independent.

Clearly an independent system is more elegant since no unnecessary assumptions are made. However, the increased difficulty of working in an independent system becomes obvious when we merely note that accepting fewer statements without proof leaves more statements to be proved. For this reason the axiomatic systems used in high-school geometry are seldom independent.

The verification that an axiomatic system is independent is also done via models. The independence of Axiom A in an axiomatic system S is established by finding a model of the system S' where S' is the system obtained from S by replacing Axiom A by a negation of A. Thus, to demonstrate that a system consisting of n axioms is independent, n models must be exhibited—one for each axiom. The independence of the axiomatic system for four-point geometry is demonstrated by the following three models, all of which interpret points as letters of the alphabet and lines as the columns of letters indicated.

Models Demonstrating Independence of Axioms for Four-Point Geometry

Model I1. A model in which a negation of Axiom 1 is true (i.e., there do not exist four points):

Points	Lines
A, *B*	*A*
	B

Since this model contains only two points, the negation of Axiom 1 is clearly true and it is easy to show that Axioms 2 and 3 are true statements in this interpretation.

Model I2. A model in which a negation of Axiom 2 is true (i.e., there are two distinct points not on one line):

Points	Lines	
A, *B*, *C*, *D*	*A*	*C*
	B	*D*

Note that in this model there is no line on points *A* and *C*. What other pairs of points fail to be on a line?

Model I3. A model in which a negation of Axiom 3 is true (i.e., there are lines not on exactly two points):

Points	Lines			
A, *B*, *C*, *D*	*A*	*A*	*B*	*C*
	B	*D*	*D*	*D*
	C			

In this model one line is on three points, whereas the remaining lines are each on two points, so the negation of Axiom 3 is true in this interpretation.

Since we have demonstrated the independence of each of the axioms of four-point geometry, we have shown that this axiomatic system is independent.

Another property that an axiomatic system may possess is completeness.

Definition 1.3. An axiomatic system is *complete* if every statement containing undefined and defined terms of the system can be proved valid or invalid, or in other words, if it is not possible to add a new independent axiom to the system.

In general, it is impossible to demonstrate directly that a system is complete. However, if a system is complete, there cannot exist two essentially different models. This means all models of the system must be pairwise *isomorphic.*

Definition 1.4. Two models α and β of an axiomatic system are said to be *isomorphic* if there exists a one-to-one correspondence ϕ from the set of points and lines of α onto the set of points and lines of β which preserves all relations. In particular if the undefined terms of the system consist of the terms "point," "line," and "incidence," then ϕ must satisfy the following conditions:

1. For each point P and line l in α, $\phi(P)$ and $\phi(l)$ are a point and line in β.
2. If P is incident with l, then $\phi(P)$ is incident with $\phi(l)$.

If all models of a system are pairwise isomorphic, it is clear that the models must each have the same number of points and lines. Furthermore, if a new independent axiom could be added to the system, there would be two distinct models of the system: a model α in which the new axiom would be valid and a model β in which the new axiom would *not* be valid. The models α and β could not then be isomorphic. Hence if all models of the system are necessarily isomorphic, it follows that the system is complete.

In the example of the four-point geometry, it is clear that models 1 and 2 are isomorphic. The verification that all models of this system are isomorphic follows readily once the following theorem is verified. (See Exercises 5 and 6.)

Theorem 1.1. *There are exactly six lines in the four-point geometry.*

Finally any discussion of the properties of axiomatic systems must include mention of the important result contained in Gödel's theorem. Greatly simplified, this result says that any consistent axiomatic system comprehensive enough to contain the results of elementary number theory is not complete.

EXERCISES

For Exercises 1–4, consider the following axiomatic system:

Axioms for Three-Point Geometry

Undefined Terms. Point, line, on.
Axiom 1. There exist exactly three points.
Axiom 2. Two distinct points are on exactly one line.
Axiom 3. Not all points are on the same line.
Axiom 4. Two distinct lines are on at least one common point.

1. (a) Prove that this system is consistent. (b) Did the proof in part (a) demonstrate absolute consistency or relative consistency? Explain.

2. Is this system independent? Why?

3. Prove the following theorems in this system: (a) Two distinct lines are on exactly one point. (b) Every line is on exactly two points. (c) There are exactly three lines.
4. Is this system complete? Why?
5. Prove Theorem 1.1.
6. Prove that any two models of four-point geometry are isomorphic.

Use the following definition in Exercises 7 and 8.

Definition. The *dual* of a statement p in the four-point geometry is obtained by replacing each occurrence of the term "point" in p by the term "line" and each occurrence of the term "line" in p by the term "point."

7. Obtain an axiomatic system for *four-line geometry* by dualizing the axioms for four-point geometry.
8. Verify that the dual of Theorem 1.1 will be a theorem of four-line geometry. How would its proof differ from the proof of Theorem 1.1 in Exercise 5?

1.3. Finite Projective Planes

As indicated by the examples in the previous section, there are geometries consisting of only a finite number of points and lines. In this section we will consider an axiomatic system for an important collection of finite geometries known as finite projective planes. These geometries may, at first glance, look much like finite versions of plane Euclidean geometry. However, there is a very important difference. In a finite projective plane, each pair of lines intersects; that is, there are no parallel lines. This pairwise intersection of lines leads to several other differences between projective planes and Euclidean planes. A few of these differences will become apparent in this section; others will not become evident until we study general plane projective geometry in Chapter 4.

Some of the first results in the study of finite projective geometries were obtained by von Staudt in 1856, but it wasn't until early in this century that finite geometries assumed a prominent role in mathematics. Since then, the study of these geometries has grown considerably and there are still a number of unsolved problems currently engaging researchers in this area.

Axioms for Finite Projective Planes

Undefined Terms. Point, line, incidence.
Defined Terms. Points incident with the same line are said to be *collinear*. Lines incident with the same point are said to be *concurrent*.

Axiom P1. There exist at least four distinct points, no three of which are collinear.
Axiom P2. There exists at least one line with exactly $n+1$ ($n > 1$) distinct points incident with it.
Axiom P3. Given two distinct points, there is exactly one line incident with both of them.
Axiom P4. Given two distinct lines there is at least one point incident with both of them.

Any set of points and lines satisfying these axioms is called a *projective plane of order n*. Note that the word "incidence" has been used as the third undefined term in this axiom system. The usage of this word rather than the word "on" is more common in the study of general projective planes.

The consistency of this axiomatic system is demonstrated by either of the following models which use the same interpretations as models 1 and 2 in Section 1.2.

Model 1

Points	Lines						
A, B, C, D, E, F, G	A	A	B	A	B	C	C
	B	D	D	F	E	D	E
	C	E	F	G	G	G	F

Model 2

Points	Lines
Dots denoted 1, 2, 3, 4, 5, 6, 7	Segments illustrated in Fig. 1.2

Note that these models are projective planes of order two and both have exactly three points on each line, but there are models with more than three points on a line as shown by the next model.

Model 3

Points	Lines												
$A, B, C, D, E,$	A	A	A	A	B	B	B	C	C	C	D	D	D
$F, G, H, I, J,$	B	E	H	K	E	F	G	E	F	G	E	F	G
K, L, M	C	F	I	L	H	I	J	I	J	H	J	H	I
	D	G	J	M	K	L	M	M	K	L	L	M	K

Whereas models 1 and 2 have three points on each line, three lines on each point, and a total of seven points and seven lines, model 3 has four points on each line, four lines on each point, and a total of 13 points and 13 lines. To