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Lecture Notes in Numerical and Applied Analysis Vol. 8

General Editors: H. Fujita and M. Yamaguti

Recent Topics in
Nonlinear PDE II

Edited by

KYÛYA MASUDA (Tohoku University)
MASAYASU MIMURA (Hiroshima University)

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RECENT TOPICS IN
NONLINEAR PDE II

PREFACE

This volume is an outgrowth of lectures delivered at the second meeting on the subject of nonlinear partial differential equations, held at Tohoku University, February 27–29, 1984: The first meeting was held at Hiroshima University, 1983. The topics presented at the conference range over various fields in mathematical physics.

We would like to take the opportunity to thank all the participants of the meeting, and the contributors to this proceedings. Special thanks should go to Professors T. Muramatsu and J. Kato who helped in many ways to make the conference a success. We are also grateful to the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan for the financial support.

K. MASUDA
M. MIMURA

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Large-Time Behavior of Viscous Surface Waves

by

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§ 1 Introduction

We are concerned with global in time solutions to a free surface problem of the viscous incompressible fluid, which is formulated as follows: The motion of the fluid is governed by the Navier-Stokes equation

$$(1.1)\qquad \left.\begin{aligned} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p &= 0 \\ \nabla\cdot u &= 0 \end{aligned}\right\} \quad \text{in } \Omega(t) ,$$

where $\Omega(t) = \{ x \in R^2 , -b < y < \eta(t,x) \}$ is the domain occupied by the fluid. The free surface $S_F : y = \eta(t,x)$ satisfies the kinematic boundary condition

$$(1.2)\qquad \eta_t + u_1\eta_{x_1} + u_2\eta_{x_2} - u_3 = 0 \qquad \text{on } S_F .$$

The stress tensor satisfies the free boundary condition :

$$(1.3)\qquad pn_i - \nu(u_{i,x_j} + u_{j,x_i})n_j = [g\eta - \beta\nabla\{(1+|\nabla\eta|^2)^{-1/2}\nabla\eta\}]n_i \qquad \text{on } S_F ,$$

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* Both authors are supported in part by the Mathematics Research Center, The University of Wisconsin-Madison.

where n is the outward normal to S_F , g is the gravitation constant and β is the nondimensionalized coefficient of surface tension. On the bottom $S_B : y = -b$ we have the fixed boundary condition

$$(1.4) \qquad u = 0 \qquad \text{on } S_B .$$

We consider the initial value problem of (1.1)-(1.4) with the data at $t = 0$

$$(1.5) \qquad \begin{cases} \eta = \eta_0(x) , & x \in R^2 , \\ u = u_0(x,y) & \text{in } \Omega_0 , \end{cases}$$

where $\Omega_0 = \Omega(0)$.

The local existence theorems for (1.1)-(1.5) are proved for both cases with or without considering the surface tension ([1],[2]). A global in time existence problem for (1.1)-(1.5) neglecting the surface tension (β=0) has a difficulty which was pointed out in [1]. However if the surface tension is taken into account, the following global existence and regularity theorem is proved.

Theorem 1.1 ([2])

Let $3 < r < 7/2$. Suppose the compatibility condition on the initial data :

$$(1.6) \qquad \begin{cases} \nabla\cdot u_0 = 0 & \text{in } \Omega_0 , \\ \{((u_0)_{i,x_j} + (u_0)_{j,x_i})n_j\}_{tan} = 0 & \text{on } y = \eta_0(x) , \\ u_0 = 0 & \text{on } y = -b . \end{cases}$$

There exists $\delta_0 > 0$ such that if the initial data satisfy

$$(1.7) \quad E_0 \equiv |\eta_0|_{H^r(R^2)} + |u_0|_{H^{r-1/2}(\Omega_0)} < \delta_0 ,$$

then there exists a unique global solution η, u, p of (1.1)-(1.5), which satisfies

$$(1.8) \quad \eta \in \tilde{K}^{r+1/2}(R^+\times R^2) , \ u \in K^r(R^+\times\Omega(t)) , \ \nabla p \in K^{r-2}(R^+\times\Omega(t)) .$$

Further given any $T_1 > 0$ and any $k > 0$, there exists $\delta_1 > 0$ such that if

$$(1.9) \quad E_0 < \delta_1$$

then the solution becomes smooth for $t > T_1$, i.e.,

$$(1.10) \quad \eta \in \tilde{K}^{r+k+1/2}((T_1,\infty)\times R^2) , \ u \in K^{r+k}((T_1,\infty)\times\Omega(t)) ,$$

$$\nabla p \in K^{r+k-2}((T_1,\infty)\times\Omega(t)) .$$

In particular the solution with $k \geq 2$ is classical. Here $H^r(\cdot)$ is the usual Sobolev space with norm $|\ |_r$ on the domain $\cdot$. $K^r((T_1,T_2)\times\Omega(t))$ is composed of the restriction to the fluid domain $\Omega(t)$ of the functions belonging to

$$(1.11) \quad K^r((T_1,T_2)\times R^3) = H^0((T_1,T_2);H^r(R^3)) \cap H^{r/2}((T_1,T_2),H^0(R^3)) .$$

$\eta \in \tilde{K}^r(R^+\times R^2)$ is defined as follows :

$\eta \in K^r((0,T)\times R^2)$ for any $T > 0$ and $\eta = \eta_1 + \eta_2$ such that $\eta_1 \in K^r(R^+\times R^2)$ and η_2 is the Fourier transform in space-time of L^1 function of bounded support.

See [2] for the detailes of the function spaces.

In this summary we give an asymptotic decay rate for the solution of the above theorem.

Theorem 1.2

If $\eta_0 \in L^1(R^2)$, then there exists $\delta_2 > 0$ such that if

$$(1.12) \qquad E_1 \equiv E_0 + |\eta_0|_{L^1} < \delta_2 \quad ,$$

then the solution has the decay rate :

$$|\partial^\alpha \eta(t)|_0 < CE_1(1+t)^{-(1+\alpha)/2} \quad , \quad \alpha = 0, 1, 2,$$

$$|u(t)|_2 \quad , \quad |\nabla p(t)|_0 < CE_1(1+t)^{-1} \quad .$$

In § 2 we transform the free boundary problem (1.1)-(1.5) to that on the fixed domain and reduce the components of the stress tensor to zero. The linear decay estimate is discussed in § 3 and the nonlinear one in § 4 .

§ 2 Reduction of the Problem

We remind ourselves some main ideas for the reduction of the free surface problem in [2]. First we use the transformation of the free boundary problem (1.1)-(1.5) to that on the fixed (equilibrium) domain : $\Omega = \{x \in R^2 , -b < y < 0\}$. Given $\eta(t,x)$ we extend it for $y < 0$ as follows :

$$(2.1) \qquad \tilde{\eta}(t,x,y) = \mathcal{F}^{-1}(e^{|\xi|y} \hat{\eta}(t,\xi)) ,$$

where $\hat{\eta}(t,\xi)$ is the Fourier transform with respect to x and $\mathcal{F}^{-1}$ is the inverse. If $\eta(t,\cdot)$ belongs to $H^s(S_F)$, then $\tilde{\eta}(t,\cdot,\cdot)$ belongs to $H^{s+1/2}(\Omega)$, where and hereafter S_F denotes the upper surface $y = 0$ of Ω . For each $t > 0$ we define the transformation θ on Ω onto $\Omega(t) = \{x \in R^2,$ $-b < y < \eta(t,x)\}$ by

$$(2.2) \qquad \theta(x_1, x_2, y; t) = (x_1,\ x_2,\ \tilde{\eta} + y(1+\tilde{\eta}/b)) \quad .$$

The vector u on $\Omega(t) = \theta(\Omega)$ is defined from the vector v on Ω by

$$(2.3) \qquad u_i = \theta_{i,x_j} v_j \;/\; J \equiv \alpha_{ij}\, v_j \quad ,$$

where J is the Jacobian determinant of $d\theta = (\theta_{i,x_j}) = 1+\tilde{\eta}/b + \tilde{\eta}_y(1+y/b)$. This map conserves the property of divergence free. $\nabla\cdot v = 0$ in Ω iff $\nabla\cdot u = 0$ in $\Omega(t)$.

Using the transformation (2.2)(2.3) and $u_{i,x_j} = \zeta_{lj}\, \partial_l(\alpha_{ik}\, v_k)$, where $\zeta = (d\theta)^{-1}$ and so on, we can rewrite the free surface problem (1.1)-(1.5) to that on the equilibrium domain Ω as follows :

$$(2.4) \qquad \eta_t - v_3 = 0 \qquad \text{on } S_F ,$$

$$\left.\begin{aligned} &(2.5) \qquad v_t - \nu\,\Delta v + \nabla q = F(\eta,\ v,\ \nabla q) \\ &(2.6) \qquad \nabla\cdot v = 0 \end{aligned}\right\} \quad \text{in } \Omega \ ,$$

$$(2.7) \qquad v = 0 \qquad \text{on } S_B \quad ,$$

$$\left.\begin{aligned} &(2.8) \qquad v_{i,x_3} + v_{3,x_i} = F_i(\eta,\ v) \quad ,\ i = 1,\ 2 \\ &(2.9) \qquad q - 2\nu\, v_{3,x_3} - (g-\beta\Delta)\eta = F_3(\eta, v) \end{aligned}\right\} \quad \text{on } S_F$$

Here we have gathered the linear terms on the left hand side and all the nonlinear terms on the right hand side of the equation.

Next we reduce the tangential component of the stress tensor

F_i , $i = 1, 2$ to zero : Given $F_i \in H^{r-3/2}(S_F)$, $i = 1, 2$, choose the vector $z \in H^{r+1}(\Omega)$ satisfying the condition

$$\begin{cases} z = 0 , \quad \partial_y z = 0 , \quad \partial_y{}^2 z = (F_2, -F_1, 0) & \text{on } S_F , \\ z = 0 , \quad \partial_y z = 0 & \text{on } S_B . \end{cases}$$

Then $w = \nabla \times z$ satisfies

$$w_3 = 0 , \quad w_{i,x_3} + w_{3,x_i} = F_i , \; i = 1, 2 \quad \text{on } S_F ,$$

$$\nabla \cdot w = 0 \qquad \text{in } \Omega ,$$

$$w_3 = 0 \qquad \text{on } S_F .$$

Therefore η, $v' = v - w$, q satisfy the system (2.4)-(2.9) with the replacements F by $F_4 = F - w_t + \nu \Delta w$ and F_i , $i = 1, 2$ by 0 . The prime of v' is omitted hereafter.

Last we rewrite the system (2.4)-(2.9) with $F = F_4$, $F_i = 0$, $i = 1, 2$, for η, v, q in the operator form. Let P be the projection on the subspace of solenoidal vectors orthogonal to the subspace $\mathcal{G}^0 = \{ \nabla \phi : \phi \in H^1(\Omega) , \phi = 0 \text{ on } S_F \}$ of $H^0(\Omega)$, i.e.,

$$(2.10) \qquad H^0 = PH^0 \oplus \mathcal{G}^0 .$$

Applying P to (2.5) we have

$$(2.11) \qquad v_t - \nu P \Delta v + P \nabla q = PF_4 .$$

Here $P \nabla q$ can be decomposed to three parts as follows :

$$P \nabla q = \nabla \pi^{(1)} + \nabla \pi^{(2)} + \nabla \pi^{(3)} ,$$

where $\pi^{(i)}$, $i = 1, 2, 3$, are defined by

$$\pi^{(1)} = 2\nu v_{3,x_3} \ , \quad \pi^{(2)} = g\eta - \beta\Delta\eta \ , \quad \pi^{(3)} = F_3 \quad \text{on } S_F \ ,$$

(2.12) $$\Delta \pi^{(i)} = 0 \quad \text{in } \Omega \ ,$$

$$\partial_y \pi^{(i)} = 0 \quad \text{on } S_B \ .$$

We denote

(2.13) $$\begin{cases} A v = -P\Delta v + \nabla\pi^{(1)} \ , \\ R v = v_3|_{S_F} \ , \\ R^*((g-\beta\Delta)\eta) = \nabla\pi^{(2)} \ . \end{cases}$$

Using these notations the system (2.4),(2.11) has the form

(2.14) $$\eta_t = R v \ ,$$

(2.15) $$v_t + A v + R^*((g-\beta\Delta)\eta) = f \ ,$$

where $f(\eta, v, \nabla q) = PF_4 - \nabla\pi^{(3)}$.

(2.6)(2.7)(2.8) with $F_i = 0$ give the domain condition of A on v .

§ 3 Rates of Decay for Linear Problem

We investigate the decay rate of the solution of the linearized equation.

(3.1) $$\eta_t = Ru \ ,$$

(3.2) $$u_t + Au + R^*((g-\beta\Delta)\eta) = 0 \ ,$$

$$(3.3) \qquad \eta(0) = \eta_0 \ , \quad u(0) = u_0 \quad \text{at} \quad t = 0 \ .$$

These are supplied with the conditions :

$$(3.4) \qquad \nabla \cdot u = 0 \quad \text{in} \ \Omega \ ,$$

$$(3.5) \qquad u_{i,x_3} + u_{3,x_i} = 0 \ , \ i = 1, 2 \quad \text{on} \ S_F \ ,$$

$$(3.6) \qquad u = 0 \quad \text{on} \ S_B \ .$$

Theorem 3.1 Let $E_2 = |\eta_0|_{L^1} + |\eta_0|_{5/2} + |u_0|_0$.

Then the solution of (3.1)-(3.6) has the decay rate :

$$(3.7) \qquad \begin{aligned} |\partial^\alpha \eta(t)|_0 &\leqq C_0 E_2 \, t^{-(1+\alpha)/2} \ , \quad 0 \leqq \alpha \leqq 5/2 \ . \\ |\ u(t)\ |_2 &\leqq C_0 E_2 \, t^{-1} \ . \end{aligned}$$

The theorem is proved by several steps.
Let $\mathcal{H} = \{ v = (\eta, u) : \eta \in H^1(S_F) \ , \ u \in PH^0(\Omega) \}$, where $(\rho, \eta)_1 = g(\rho, \eta)_0 + \beta(\nabla\rho, \nabla\eta)_0$ is the inner product of $H^1(S_F)$ and set $W = \{ v : \eta \in H^{5/2}(S_F) \ , \ u \in PH^2(\Omega)$ and u satisfies (3.4)(3.5)(3.6) $\}$. Let us define the operator

$$(3.9) \qquad G\,v = \begin{pmatrix} 0 & R \\ -R^*(g-\beta\Delta)\eta & -A \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix} \quad \text{from } D(G) \text{ in } \mathcal{H} \ ,$$

and consider its closed extension which will be denoted by G again.

Lemma 3.2

The operator G generates a contraction semigroup e^{tG} on $\mathcal{H}$, and $W \subset D(G)$.

Consider the resolvent equation :

$$(3.10) \qquad (\lambda - G)\begin{pmatrix} \eta \\ u \end{pmatrix} = \begin{pmatrix} h \\ f \end{pmatrix}$$

The resolvent of G can be extended to the left half plane as lemmas 3.3-3.5.

Lemma 3.3 For any $\tau_0 > 0$ there exists $c_0 > 0$ such that if $\lambda \in \{ \lambda = \sigma + i\tau , -c_0|\tau| < \sigma < \tau_0 , |\tau| > \tau_0 \}$, then the solution of (3.10) has the estimate

$$(3.11) \qquad |u|_2 + |\lambda||u|_0 + |\lambda^{-1}Ru|_{5/2} + |\eta|_{5/2} + |\lambda||\eta|_{3/2} \leqq C(|f|_0 + |h|_{5/2}) .$$

We treat the resolvent near $\lambda = 0$ in two cases separately, i.e., (i) The supports of $\hat{h}(\xi)$, $\hat{f}(\xi,y)$ belong to $\{|\xi| \geq \xi_0\}$. (ii) The supports belong to $\{|\xi| \leq \xi_0\}$. Here ^ means the Fourier transform with respect to x.

Lemma 3.4

For any $\xi_0 > 0$ there exists $r_0 > 0$ such that if $\lambda \in \{ |\lambda| < r_0 \}$ and the supports of $\hat{h}(\xi)$, $\hat{f}(\xi,y)$ belong to $\{ |\xi| \geq \xi_0 \}$, then the resolvent equation (3.10) has the solution (η, u) satisfying

$$(3.12) \qquad |u|_2 , |\eta|_{5/2} \leq C (|h|_{5/2} + |f|_0)$$

Let $\hat{G}(\xi)$ be the Fourier transform of G with respect to x.

Lemma 3.5

There exist $\xi_1 > 0$ and r_1, r_2 ($\nu(\pi/2b)^2 > r_2 > r_1 > 0$) such that if $r_1 < |\lambda| < r_2$ and $|\xi| < \xi_1$, then $(\lambda - \hat{G}(\xi))^{-1}$ exists except for a one-dimensional eigenspace which is analytic with respect to ξ. The eigenvalue and eigenvector have the following expansions.

$$(3.13)\quad \begin{cases} \lambda = -(gb^3/3\nu)|\xi|^2 + O(|\xi|^4) , \\ \eta = 1 + O(|\xi|^2) , \\ u_j = i(g\xi_j/2\nu)(y^2-b^2) + O(|\xi|^3) , \quad j = 1, 2 , \\ u_3 = (g|\xi|^2/2\nu)(y^3/3-b^2y-2b^3/3) + O(|\xi|^4) . \end{cases}$$

By using lemmas 3.2-3.5 the decay estimate (3.7) can be proved by the transformation of the integral path of the reprensentation

$$(3.14)\quad v(t) = e^{tG} v_0 = \lim_{\tau\to\infty} \frac{1}{2\pi i}\int_{\sigma-i\tau}^{\sigma+i\tau} e^{\lambda t} (\lambda - G)^{-1} v_0 \, d\lambda , \quad \sigma > 0$$

to the left half plane.

§ 4 Nonlinear Decay Estimates

The free surface problem (1.1) - (1.5) was reduced to the following system in § 2.

$$(4.1)\quad \eta_t - R u = 0 ,$$

$$(4.2)\quad u_t + A u + R^*((g-\beta\Delta)\eta) = f ,$$

$$(4.3)\quad \eta(0) = \eta_0 , \quad u(0) = u_0 ,$$

where f is nonlinear terms depending on η, u, ∇p and their derivatives.

The initial value problem (4.1) - (4.3) has the unique solution (Theorem 1.1) which becomes smooth for $t \geqq T_1 > 0$. Namely we know that if $E_0 < \delta_1$, then