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Mappings of Operator Algebras

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Shôichirô Sakai

This is dedicated to Shôichirô Sakai, for his brilliant solutions to many of the key problems in the theory of operator algebras that gave us the means and the courage to move ahead and for his inspired leadership in the development of the crucially important theory of unbounded derivations.

With Gratitude and Respect from the Participants of the U.S.–Japan Joint Seminar on Mappings of Operator Algebras

"The results which we shall obtain throw light on an entirely new side of operator theory."

F.J. Murray and J. von Neumann, 1936

Preface

This volume consists of articles contributed by participants at the fourth Japan–U.S. Joint Seminar on Operator Algebras. The seminar took place at the University of Pennsylvania from May 23 through May 27, 1988 under the auspices of the Mathematics Department. It was sponsored and supported by the Japan Society for the Promotion of Science and the National Science Foundation (USA). This sponsorship and support is acknowledged with gratitude.

The seminar was devoted to discussions and lectures on results and problems concerning mappings of operator algebras (C^*-and von Neumann algebras). Among the articles contained in these proceedings, there are papers dealing with actions of groups on C^* algebras, completely bounded mappings, index and subfactor theory, and derivations of operator algebras.

The seminar was held in honor of the sixtieth birthday of Shôichirô Sakai, one of the great leaders of Functional Analysis for many decades. This volume is dedicated to Professor Sakai, on the occasion of that birthday, with the respect and admiration of all the contributors and the participants at the seminar.

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On Convex Combinations of Unitary Operators in C^*-Algebras

UFFE HAAGERUP

1. INTRODUCTION

Let A be a unital C*-algebra. In [1], Gardner proved that if $x \in A$ and $\|x\| < 1$, then

$$x + U(A) \subseteq U(A) + U(A),$$

where $U(A)$ is the set of unitaries in A. Kadison and Pedersen discovered in [2] that this inclusion together with a simple inductive argument led to the following result:

Let $n \in \mathbb{N}$, $n \geq 3$. If $y \in A$ and $\|y\| < 1 - \frac{2}{n}$, then y is of the form

$$y = \tfrac{1}{n}(u_1 + \ldots + u_n)\ , \quad u_i \in U(A)\ ,$$

i.e: y is a mean of n unitaries in A. The main result of this paper is

Theorem

Let A be a unital C^*-algebra and let $n \in \mathbb{N}$, $n \geq 3$. If $y \in A$ and $\|y\| \leq 1 - \frac{2}{n}$, then

$$y = \tfrac{1}{n}(u_1 + \ldots + u_n)$$

for some $u_1, \ldots, u_n \in U(A)$.

This settles a conjecture of Olsen and Pedersen in [3]. The result is best possible for general C^*–algebras, because if y is a scalar multiple of a non–unitary isometry in a C^*–algebra A and $\|y\| > 1 - \frac{2}{n}$, then y is not the mean of n unitaries in A (cf. [2]). For special C^*–algebras much stronger results hold. Rørdam proved in [5] that if the invertible operators are dense in A, then every operator x in the closed unit ball of A is the mean of n unitaries in A for any $n \geq 3$.

The above Theorem is well known and easy to prove for $n = 4$, and it was proved for $n = 3$ in [3]. Our proof of the general case is obtained by proving the following analogue of Gardner's inclusion:

If $x \in A$ and $\|x\| \leq 1$, then

$$x + 2P \subseteq U(A) + 2P$$

where $P = \{uh \mid u \in U(A), h \in A_+, \|h\| \leq 1\}$.

In the last section of the paper we study the case $n = 3$ more closely. It is proved that for $n = 3$, u_1, u_2 and u_3 can be chosen, such that the spectra of $u_1^*u_2$, $u_2^*u_3$ and $u_3^*u_1$ are contained in the semicircle $\{e^{i\nu} | 0 \leq \nu \leq \pi\}$. Moreover, if $\|y\| < \frac{1}{3}$, then $u_1 \in U(A)$ can be chosen freely and u_2, u_3 are then uniquely determined by the spectral conditions above. This implies that for $x \in A$, $\|x\| < 1$, there is a homeomorphism Φ_x of $U(A)$, such that $\Phi_x^3 = \mathrm{id}$ and

$$x = u + \Phi_x(u) + \Phi_x^2(u)$$

for all $u \in U(A)$.

We wish to thank Gert K. Pedersen for suggesting a major simplification of the proof of the spectral conditions in the key lemma (Lemma 1), which reduced the length of the proof from 8 to 2 manuscript pages.

2. THE MAIN RESULT

Lemma 1

Let A be a unital C*–algebra. Let $x, h \in A$ $\|x\| \le 1$, $\|h\| \le 1$, $h \ge 0$. Then

$$x + 2h = u_1 + u_2 + u_3$$

where $u_1,u_2,u_3 \in U(A)$ and $sp(u_1^*u_2)$, $sp(u_2^*u_3)$ are both contained in $\{e^{i\nu} \mid -\frac{\pi}{2} \le \nu \le \pi\}$.

Proof. Write $x = a + ib$, where $a,b \in A_{sa}$. Then

$$a^2 + b^2 = \frac{1}{2}(x^*x + xx^*) \le 1.$$

Particularly $\|a\| \le 1$ and $\|b\| \le 1$. By [4, Prop. 1.3.8] the function $t \to t^{\frac{1}{2}}$ is operator monotone on $[0,\infty[$. Therefore

$$|a| \le (1 - b^2)^{\frac{1}{2}}.$$

Hence

$$-(1 - b^2)^{\frac{1}{2}} \le a \le (1 - b^2)^{\frac{1}{2}}.$$

Put

$$c = \frac{1}{2}(2h + a - (1 - b^2)^{\frac{1}{2}}).$$

Then

$$h - (1 - b^2)^{\frac{1}{2}} \le c \le h.$$

Particularly $\|c\| \le 1$, and $c + (1 - b^2)^{\frac{1}{2}} \ge 0$.

Put

$$u_1 = c - i(1 - c^2)^{\frac{1}{2}}$$

$$u_2 = (1 - b^2)^{\frac{1}{2}} + ib$$

$$u_3 = c + i(1 - c^2)^{\frac{1}{2}} .$$

Then $u_1,u_2,u_3 \in U(A)$ and

$$u_1 + u_2 + u_3 = x + 2h.$$

(For $h = 0$, this choice of u_1,u_2,u_3 was used in the proof of [3, Thm. 4.3]).

To prove the conditions on the spectra of $u_1^*u_2$ and $u_2^*u_3$ stated in the lemma, we have to prove that if $\lambda = \cos\theta + i\sin\theta$, $0 < \theta < \frac{\pi}{2}$, then

$$-\lambda \notin \mathrm{sp}(u_1^*u_2) \text{ and } -\lambda \notin \mathrm{sp}(u_2^*u_3).$$

Let λ be as above and assume that $-\lambda \in \mathrm{sp}(u_1^*u_2)$. Then for some faithful representation of A on a Hilbert space H, there is a unit vector $\xi \in H$ for which

$$u_1^*u_2\xi = (-\lambda)\xi.$$

(cf. [4, 4.3.10]). Thus $(u_2 + \lambda u_1)\xi = 0$. Let φ be the vector state on A given by ξ. Then $\varphi(u_2 + \lambda u_1) = 0$, or equivalently

$$\varphi((1-b^2)^{\frac{1}{2}} + ib + (\cos\theta + i\sin\theta)(c - i(1-c^2)^{\frac{1}{2}}) = 0.$$

By considering the real part, we get:

$$\varphi((1-b^2)^{\frac{1}{2}} + \cos\theta\, c + \sin\theta\,(1-c^2)^{\frac{1}{2}}) = 0.$$

By rewriting the equation in the form

$$\cos\theta\,\varphi((1-b^2)^{\frac{1}{2}} + c) + (1-\cos\theta)\varphi((1-b^2)^{\frac{1}{2}})$$
$$+ \sin\theta\,\varphi((1-c^2)^{\frac{1}{2}}) = 0$$

and using $(1-b^2)^{\frac{1}{2}} + c \geq 0$, $(1-b^2)^{\frac{1}{2}} \geq 0$, $(1-c^2)^{\frac{1}{2}} \geq 0$, $0 < \theta < \frac{\pi}{2}$ one gets

$$\varphi((1-b^2)^{\frac{1}{2}} + c) = \varphi((1-b^2)^{\frac{1}{2}}) = \varphi((1-c^2)^{\frac{1}{2}}) = 0.$$

If $(s\xi,\xi) = 0$ for some positive operator $s \in B(H)$, then also $s\xi = 0$. Therefore

$$((1-b^2)^{\frac{1}{2}} + c)\xi = (1-b^2)^{\frac{1}{2}}\xi = (1-c^2)^{\frac{1}{2}}\xi = 0.$$

Particularly $c\xi = (1-c^2)^{\frac{1}{2}}\xi = 0$. This is a contradiction, because

$$\|c\xi\|^2 + \|(1-c^2)^{\frac{1}{2}}\xi\|^2 = 1.$$

The spectral condition on $u_2^*u_3$ can be proved as above by using that

$$sp(u_2^*u_3) = sp(u_3u_2^*).$$

Indeed, if $\lambda = \cos\theta + i\sin\theta$, $0 < \theta < \frac{\pi}{2}$ and $-\lambda \in sp(u_3u_2^*)$, then we can find a vector state φ on A, such that

$$\varphi(u_2^* + \lambda u_3^*) = 0$$

and by taking the real part we get

$$\varphi((1-b^2)^{\frac{1}{2}} + \cos\theta\, c + \sin\theta(1-c^2)^{\frac{1}{2}}) = 0,$$

which is the same equality as in the previous case.

Lemma 2

Let A be a unital C*–algebra and let

$$P = \{uh \mid u \in U(A),\ h \in A_+,\ \|h\| \leq 1\}.$$

Then for every $x \in A$, $\|x\| \leq 1$ one has

$$x + 2P \subseteq U(A) + 2P.$$

Proof. Since the spectral condition in Lemma 1 is invariant under multiplication from left (and right) with a fixed unitary, it follows that every $z \in x+2P$ is of the form

$$z = u_1 + u_2 + u_3$$

with $sp(u_1^*u_2)$ and $sp(u_2^*u_3)$ contained in $F = \{e^{i\nu} \mid -\frac{\pi}{2} \le \nu \le \pi\}$. The branch of the square root given by $e^{i\nu} \to e^{i\nu/2}$, $-\frac{\pi}{2} \le \nu \le \pi$, is continuous on F. Hence there exists $v \in U(A)$ such that

$$u_2^*u_3 = v^2$$

and

$$sp(v) \subseteq \{e^{i\nu} \mid -\tfrac{\pi}{4} \le \nu \le \tfrac{\pi}{2}\}.$$

Put $w = u_3v^* = u_2v$. Then

$$u_2 + u_3 = w(v + v^*),$$

and since $sp(v) \subseteq \{s \mid \mathrm{Re}\, s \ge 0\}$, we have $v + v^* \ge 0$, so that $u_2 + u_3 \in 2P$. This proves lemma 2.

Theorem

Let A be a unital C*–algebra, and let $n \in \mathbb{N}$, $n \ge 3$. If $y \in A$ and $\|y\| \le 1 - \frac{2}{n}$, then

$$y = \tfrac{1}{n}(u_1 + \ldots + u_n)$$

for some $u_1,\ldots,u_n \in U(A)$.

Proof. Put $x = \frac{n}{n-2}\, y$. Then $\|x\| \le 1$. By repeated use of lemma 2, we get

$$(n-2)x + 2P \subseteq \underbrace{U(A) + \ldots + U(A)}_{n-2 \text{ times}} + 2P$$

But $2P \subseteq U(A) + U(A)$, because every selfadjoint operator h of norm less or equal to 1 is the half sum of the two unitaries $h \pm i(1-h^2)^{\frac{1}{2}}$. Hence

$$(n-2)x + 2P \subseteq \underbrace{U(A) + \ldots + U(A)}_{n \text{ times}}.$$

Since $0 \in 2P$, we get

$$y = \frac{n-2}{n} x \in \frac{1}{n}(\underbrace{U(A) + \dots + U(A)}_{n \text{ times}}).$$

3. ON THE SUM OF THREE UNITARIES

Proposition 1

Let A be a unital C*–algebra, and let $x \in A$, $\|x\| \leq 1$. Then

$$x = u_1 + u_2 + u_3 ,$$

where $u_1,u_2,u_3 \in U(A)$, and $sp(u_1^*u_2)$, $sp(u_2^*u_3)$, $sp(u_3^*u_1)$ are contained in $\{e^{i\nu} | 0 \leq \nu \leq \pi\}$.

Proof. Let u_1,u_2,u_3 be as in lemma 1 with $h = 0$, i.e.

$$x = a + ib, \quad a,b \in A_{s.a.}$$

$$c = \tfrac{1}{2}(a - (1-b^2)^{\frac{1}{2}})$$

and

$$(*) \quad \begin{cases} u_1 = c - i(1-c^2)^{\frac{1}{2}} \\ u_2 = (1-b^2)^{\frac{1}{2}} + ib \\ u_3 = c + i(1-c^2)^{\frac{1}{2}} \end{cases}$$

Recall from the proof of Lemma 1 that $a \leq (1-b^2)^{\frac{1}{2}}$. Hence $c \leq 0$, and therefore $sp(u_1)$ is contained in

$$\{- e^{i\nu} | 0 \leq \nu \leq \tfrac{\pi}{2}\}.$$

This implies that

$$\mathrm{sp}(u_3^*u_1) = \mathrm{sp}(u_1^2) \subseteq \{e^{i\nu} \mid 0 \le \nu \le \pi\}.$$

To prove that $\mathrm{sp}(u_1^*u_2)$ and $\mathrm{sp}(u_2^*u_3)$ are also contained in $\{e^{i\nu} \mid 0 \le \nu \le \pi\}$ we can proceed as in the proof of lemma 1. From lemma 1 we already know that the two spectra are contained in $\{e^{i\nu} \mid -\frac{\pi}{2} \le \nu \le \pi\}$, so it remains to be proved that if $\lambda = \cos\theta + i\sin\theta$ for $\frac{\pi}{2} \le \theta < \pi$, then

$$-\lambda \notin \mathrm{sp}(u_1^*u_2) \text{ and } -\lambda \notin \mathrm{sp}(u_2^*u_3).$$

As in the proof of prop. 1, $-\lambda \in \mathrm{sp}(u_1^*u_2)$ or $-\lambda \in \mathrm{sp}(u_2^*u_3)$ would imply that

$$(*) \qquad \varphi((1-b^2)^{\frac{1}{2}} + \cos\theta\, c + \sin\theta\,(1-c^2)^{\frac{1}{2}}) = 0.$$

Since $(1-b^2)^{\frac{1}{2}} \ge 0$, $(1-c^2)^{\frac{1}{2}} \ge 0$, $c \le 0$, $\cos\theta \le 0$ and $\sin\theta > 0$ it follows that

$$\varphi((1-b^2)^{\frac{1}{2}}) = \cos\theta\, \varphi(c) = \sin\theta\, \varphi(1-c^2)^{\frac{1}{2}} = 0.$$

Particularly

$$\varphi((1-b^2)^{\frac{1}{2}}) = \varphi((1-c^2)^{\frac{1}{2}}) = 0.$$

Since $-(1-b^2)^{\frac{1}{2}} \le a \le (1-b^2)^{\frac{1}{2}}$, we have

$$0 \le -c \le (1-b^2)^{\frac{1}{2}}.$$

Hence also $\varphi(-c) = 0$. We can assume that φ is a vector state $(\cdot\ \xi,\xi)$ for faithful representation of A on a Hilbert space. Then

$$(1-c^2)^{\frac{1}{2}}\xi = (-c)\xi = 0,$$

because $(1-c^2)^{\frac{1}{2}} \ge 0$ and $(-c) \ge 0$. This contradicts that

$$\|c\xi\|^2 + \|(1-c^2)^{\frac{1}{2}}\xi\|^2 = 1.$$

Hence $\mathrm{sp}(u_1^*u_2)$ and $\mathrm{sp}(u_2^*u_3)$ are both contained in $\{e^{i\nu} \mid 0 \le \nu \le \pi\}$.