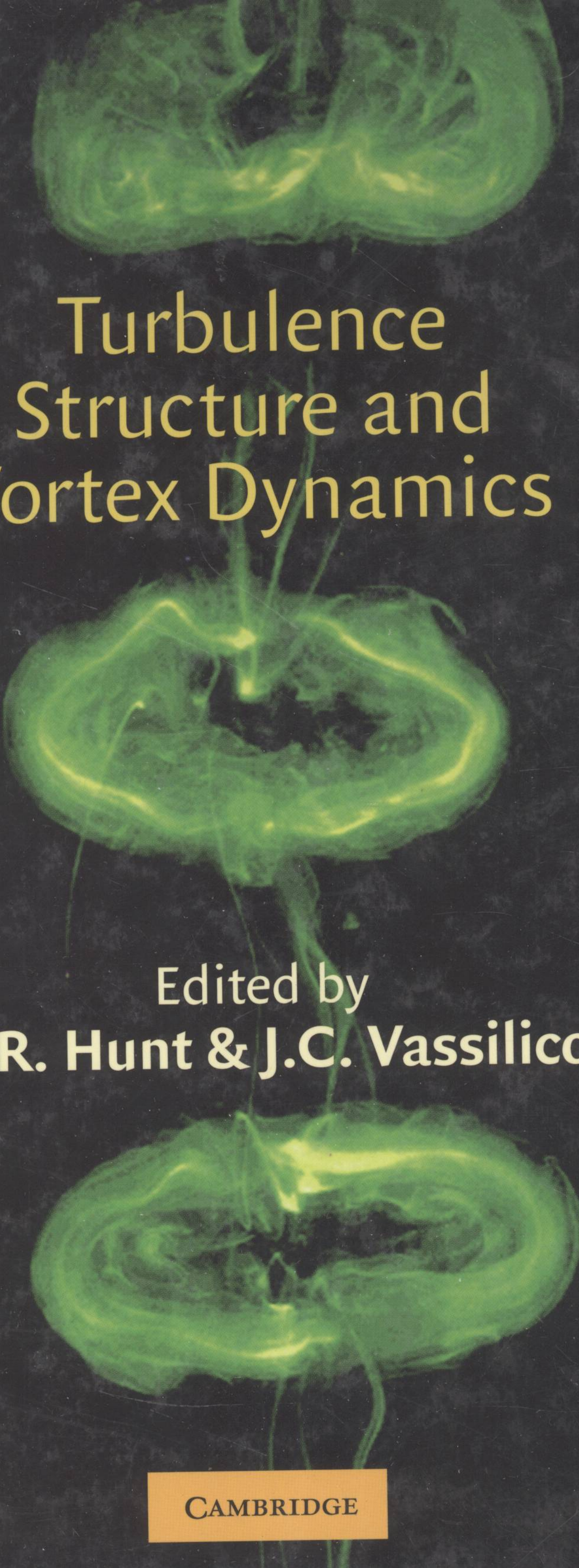

Turbulence Structure and Vortex Dynamics

Edited by

J.C.R. Hunt & J.C. Vassilicos

CAMBRIDGE

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The Isaac Newton Institute of Mathematical Sciences of the University of Cambridge exists to stimulate research in all branches of the mathematical sciences, including pure mathematics, statistics, applied mathematics, theoretical physics, theoretical computer science, mathematical biology and economics. The research programmes it runs each year bring together leading mathematical scientists from all over the world to exchange ideas through seminars, teaching and informal interaction.

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Introduction

Leonardo da Vinci's drawings of eddies below waterfalls, John Constable's paintings of swirling and disintegrating cloud shapes and L.F. Richardson's Swiftian rhyme all show different aspects of the essential nature of turbulence. When expressed in prosaic scientific language the modern understanding of turbulence is that it is a collection of weakly correlated vortical motions, which, despite their intermittent and chaotic distribution over a wide range of space and time scales, actually consist of local characteristic 'eddy' patterns that persist as they move around under the influences of their own and other eddies' vorticity fields. Numerical simulations and experimental observations have now identified basic forms and even the 'life-cycles' of some of these structures. Some of them, for example, seem to appear as local shear layers, then evolve into vortex tubes and finally break up. In some cases quite extreme distortion and interaction between vortices lead to very large local velocities. These universal features occur in all highly turbulent flows. However, because the largest scale eddies extend across the whole flow and are strongly influenced by the boundary conditions they are not universal; nevertheless they tend to have the same characteristic forms in each type of turbulent flow.

In the Isaac Newton Institute (INI) programme on turbulence held between January and July 1999 there were several workshops and conferences on different aspects of the subject. All of them succeeded in bringing together physicists, engineers, mathematicians and experimentalists, as can be seen in this and other volumes and review articles describing the programme (Voke, Sandham & Kleiser 1999; Launder & Sandham 2000; Vassilicos 2000; Hunt, Sandham, Vassilicos, Launder, Monkewitz & Hewitt 2000).

In the Symposium on Vortex Dynamics and Turbulence Structure there were lectures and discussions on a number of key questions that have engaged turbulence researchers for many years. What is the overall significance for turbulent flows of vortical structures? How should one study their persistence and characteristic structure; do they correspond to some kind of eigensolutions of the basic equations or of some reduced form of these equations; what are their geometrical statistics and their stability, given that they exist in a chaotic environment with many other structures surrounding them? How do they interact or not interact with each other and with surrounding turbulence, and what are their dissipative properties? Are the near-singularities of the turbulence or the conjectured finite-time singularities related to the vortical or other (e.g. straining) structures, and if so what kind? What are the Eulerian and Lagrangian properties of such structures, and how do their conditional statistics relate to the well-established unconditional Eulerian and

Lagrangian statistics (e.g. spectra, energy cascades up- and down-scale, relative motions of particles) and the scaling properties of the entire flow? To what extent can turbulence be represented in terms of space-filling functions such as Fourier or Chebychev basis functions or is it necessary to work in terms of localised functions such as wavelets.

The articles in this volume address all these questions. Most involve mathematical analysis, but some describe numerical simulations and experimental results that focus on these questions. Some of the papers focused on the deterministic kinds of vortical motion that characterise eddy motions, while others also relate these studies to the overall statistics of the turbulent flows which can be measured more readily than the details of individual eddies. Only one paper is exclusively concerned with the statistical dynamics of turbulence.

Deterministic analyses were applied to isolated vortices, to their response when subject to large scale rotational and irrotational straining, and to their interaction with each other. In some situations large scale straining is a reasonable 'mean-field' approximation for the average effects of all other vortices. But in other situations it is necessary to consider specific interactions between small numbers of vortices. **Fukumoto & Moffatt** analyse the effect of viscosity on the motion of a vortex ring, and how the diffusion of vorticity changes its motion. The straining of vortices are considered in three papers; **Gibbon, Galanti & Kerr** consider the general mathematical properties of the stretching and compression of vorticity, including the surprising fact that its tendency to become a singularity at any point in the flow is related to the overall properties of the flow.

There are many different ways that finite amplitude vortices can be stretched and distorted, and **Le Dizes** presents an analysis of a new family of stretched non-axisymmetric vortices. As elongated vortices are stretched and distorted by external straining fields, oscillations and waves can develop and lead to the formation of new structures and ultimately to the total breakdown into small scale chaotic motion. The basic mechanisms of these 'core dynamics' are reviewed by **Pradeep & Hussain**. In some cases the external motions are caused by adjacent vortices and then the instability and transformations are coupled in a global sense, as shown in the experimental paper of **Williamson, Leweke & Miller**. In 'classical' fluids such as air and water at ambient temperature, the vorticity in a vortex diffuses out of vortices or is exchanged when vortices interact as a result of molecular diffusion. In superfluids at very low temperatures these diffusion processes do not occur and therefore vortices move and interact with each other according to the theory of ideal inviscid flow. However certain quantum effects also lead to dissipative phenomena such as reconnection. This is the motivation of **Barenghi's** paper on ideal fluid turbulence and its relation to normal fluid turbulence.

Other papers here show how a combination of deterministic and statistical analyses of turbulent velocity fields is leading to a better understanding of the qualitative characteristics of the eddy motion in turbulence as well as to quantitative predictions. Much research is based on the assumption that this is the key to improving the approximate models of turbulence (such as Large Eddy Simulation and spectral models) and to assessing their accuracy and range of application. **Leonard** analyses, following the earlier ideas of Synge & Lin (1943), the dynamics and kinematics of small individual eddies or packets of vorticity, strained by eddy motions with larger length scales. He explores the limits when the lengths of the strained eddies become comparable with the larger ones, and tend to form elongated and randomly twisted 'ribbons'. The consequences for the spectra are worked out.

Novikov explains why this dynamical interaction implies that small scale turbulence may not be as statistically independent of the large scales as is assumed in Kolmogorov's theory; there may be fewer degrees of freedom and some aspects of their motion may be 'slaved' to the larger scales on some 'slow' manifold. He derives some statistical conditions based on this concept. However the eddy motions do need to be considered because they determine the intermittency of turbulence which he explains as being crucial to the interpretation of the overall turbulence statistics.

Warhaft's discussion of experimental measurements of small scale turbulence also takes up this theme. The higher the order of the statistical moments the more they are anisotropic. These are associated with small scale organised structures, in which there are strong local gradients in both the velocity and scalar fields,. He demonstrates that the structures can be defined more precisely if measurements are made at three rather than two points simultaneously, which has been usual up to now.

Flow visualisation and experiments have indicated that these structures are quite geometrically complex, often approximating to sheets of vorticity and scalars wound up into spiral forms which correspond to a type of ideal mathematical singularity. **Vassilicos** analyses such velocity fields and their effects on the diffusion of scalars; he also shows how these types of eddy can be detected when they occur at random positions in numerical simulations of turbulent flows. He demonstrates how such structures are consistent with the 'anomalous' scaling laws found in statistical correlations in fully developed turbulence.

Tsinober analyses the dynamical equations governing correlations between the straining and vorticity fields of small scale turbulence, in order to clarify the relative roles of vortex stretching and straining, or relative advection, in producing even smaller scales and thence dissipation. His results suggest that it may be necessary to consider a cycle of stretching and straining of eddy motions to understand the full dynamics; indeed the simple, rather static concept of vortex stretching is quite inconsistent with the production of smaller scales. Like Betchov in 1956 he has the temerity to propose an amendment to L.F. Richardson's rhyme about the roles of great whirls and lesser whirls in the cascade process!

Hunt's paper is similar to Leonard's in assuming that the analysis of the non-linear interactions in turbulence can be usefully idealised as a sequence of events when small scale vortices are strained by large scale motions. He discusses how the weakly non-linear effects cause the vortex sheets to roll up, or become unstable. Curiously there is a geometrical problem to be solved: how to define the changes in these shapes, which are associated with the cycle of growth, transformation and breakup of small scale eddies, that Tsinober analyses using statistical data in his paper. Hunt also reviews an outstanding kinematical question about turbulence as to when and to what extent spectra reflect on the one hand the forms of the eddies themselves, especially their singularities, and on the other, the distribution of their amplitudes with wavenumber (or frequency).

Cambon takes up the question, touched on by Hunt, that when vortices are formed in turbulence, for example as a result of straining by larger scales, various kinds of waves and instabilities tend to grow. He reviews and relates a number of current mathematical techniques used for analysing these perturbations. He points out how some are local and some global; some are based on eigen solutions, while others are based on general linear solutions more dependent on initial conditions. Many interesting special cases are described in detail, and reasons are given why cyclonic eddies are more stable than anticyclonic.

In numerical simulations the resolution is now fine enough for even small scale flow structure to be described for high Reynolds number turbulence. **Lesieur, Comte and Metais** use Large Eddy Simulation techniques to examine the structure of the vortices that form in shear flows and rotating flows. They explain how the vortices contribute to the statistical distribution of kinetic energy in the turbulence, as well as describing in some detail how the different scales and orientations of vortices are related in these chaotic flows, which have a high degree of local organisation.

On long enough timescales it is likely that the internal eddy structure is unimportant, in which case turbulence can be analysed rather like a visco-elastic fluid, based on the concepts and methods of statistical physics. **McComb & Johnston** use methods involving the Renormalisation Group. In conjunction with novel assumptions about the statistical independence of the small eddy scales, they derive quantitatively the energy spectrum of turbulence and new results about the internal 'eddy viscosity' that controls the energy transfer between eddy scales. These methods may well have wider applications to more complex flows in future.

We, and we believe all the speakers at the workshop, are extremely grateful

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References

Hunt, J.C.R., Sandham, N., Vassilicos, J.C., Launder, B.E., Monkewitz, P.A. & Hewitt, G.F., 2000, 'Developments in turbulence research: a review based on the 1999 Programme of the Isaac Newton Institute, Cambridge', submitted to *J. Fluid Mech.*

Launder, B.E. & Sandham, N.D. (eds.), 2000, *Closure Strategies for Turbulent and Transitional Flows*, Cambridge University Press.

Synge, J.L. & Lin, C.C., 1943, 'On a statistical model of isotropic turbulence', *Trans. R. Soc. Canada*, **37**, 45–79.

Vassilicos, J.C., (ed.), 2000, *Intermittency in Turbulent Flows*, Cambridge University Press.

Voke, P.R., Sandham, N., & Kleiser, L. (eds.), 1999, *Direct Large-Eddy Simulation III*, Proceedings of the Isaac Newton Institute Symposium/ERCOFTAC Workshop, Cambridge UK, 12–14 May 1999, ERCOFTAC Series, Vol. 7, Kluwer Academic Publishers.

Julian Hunt
Christos Vassilicos

CONTENTS

Motion and Expansion of a Viscous Vortex Ring: Elliptical Slowing Down and Diffusive Expansion

Yasuhide Fukumoto and H.K. Moffatt

1 Introduction

The motion of a vortex ring is a venerable problem, and, since the attempts of Helmholtz and Kelvin in the last century, extensive study has been made on various dynamical aspects, such as formation, traveling speed, waves, instability, interactions and so on. Concerning the steady solution for inviscid dynamics, analytical technique has been matured enough to make a highly nonlinear regime tractable. In contrast, the effect of viscosity on the nonlinear dynamics is poorly understood even for an isolated vortex ring.

In this article, we present a large-Reynolds-number asymptotic theory of the Navier–Stokes equations for the motion of an axisymmetric vortex ring of small cross-section. Our intention is to make the nonlinear effect amenable to analysis by constructing a framework for calculating higher-order asymptotics. The nonlinearity is featured by deformation of the core cross-section. We build a general formula for the translation speed incorporating the slowing-down effect caused by the elliptical deformation of the core. Moreover we show that viscosity has the action of expanding the ring radius, simultaneously with swelling the core; starting from an infinitely thin circular loop of radius R_0, the radii $R_s(t)$ of the loop of stagnation points relative to a comoving frame, $R_p(t)$ of the loop of peak vorticity, $R_c(t)$ of the centroid of vorticity all grow linearly in time t as $R_s \approx R_0 + 2.5902739\nu t/R_0$, $R_p \approx R_0 + 4.5902739\nu t/R_0$, and $R_c \approx R_0 + 3\nu t/R_0$. It is pointed out that the asymptotic values of R_p and R_c exhibit a discrepancy, at a finite Reynolds number, from the numerical result of Wang, Chu & Chang (1994).

To begin with, we briefly survey known results. Dyson (1893) (see also Fraenkel 1972) extended Kelvin's formula for the speed U of a thin axisymmetric vortex ring, steadily translating in an inviscid incompressible fluid of infinite extent, to third (virtually fourth) order in a small parameter $\varepsilon = \sigma/R_0$, the ratio of core radius σ to the ring radius R_0, as

$$U = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8}{\varepsilon}\right) - \frac{1}{4} - \frac{3\varepsilon^2}{8}\left[\log\left(\frac{8}{\varepsilon}\right) - \frac{5}{4}\right] + O(\varepsilon^4 \log\varepsilon)\right\}, \qquad (1.1)$$

where Γ is the circulation carried by the ring. The vorticity is assumed to be in proportion to distance from the axis of symmetry. We consider Kelvin's formula (the first two terms) as the first-order and the $O(\varepsilon^2)$-terms as the third. The local self-induced flow consists not only of a uniform flow but also of a straining field. The latter manifests itself at $O(\varepsilon^2)$ and deforms the core into an ellipse, elongated in the propagating direction:

$$r = \sigma \left\{ 1 - \frac{3\varepsilon^2}{8} \left[\log\left(\frac{8}{\varepsilon}\right) - \frac{17}{12} \right] \cos 2\theta + \cdots \right\}, \tag{1.2}$$

where (r, θ) are local moving cylindrical coordinates about the core center which will be introduced in §2. The inclusion of the third-order term in the propagating velocity gives a remarkable improvement in approximation; (1.1) compares well even with the exact value for the 'fat' limit of Hill's spherical vortex (Fraenkel 1972). In this limit, the parameter ε is as large as $\sqrt{2}$ under a suitable normalisation. This surprising agreement encourages us to explore a higher-order approximation in more general circumstances.

Viscosity acts to diffuse vorticity, and the motion ceases to be steady. Its influence on the traveling speed, at large Reynolds number, was first addressed by Tung & Ting (1967), using the matched asymptotic expansions, for the case where the the vorticity is, at a virtual instant $t = 0$, a 'δ-function' concentrated on a circle of radius R_0. By a different method, Saffman (1970) succeeded in deriving an explicit formula, valid up to first order in $\epsilon \equiv (\nu/\Gamma)^{1/2}$, as

$$U = \frac{\Gamma}{4\pi R_0} \left[\log\left(\frac{8R_0}{2\sqrt{\nu t}}\right) - \frac{1}{2}(1 - \gamma + \log 2) + \cdots \right], \tag{1.3}$$

where ν is the viscosity, t is the time, and $\gamma = 0.57721566\cdots$ is Euler's constant (see also Callegari & Ting 1978). Wang, Chu & Chang (1994) employed a similar method to Tung & Ting (1967), but with a different choice $\sqrt{t}$ as small parameter, and gained a correction to (1.3) originating from the viscous diffusive effect. This correction vanishes in the limit of $\nu \longrightarrow 0$. Unfortunately, the existing asymptotic theories all assume a circular symmetric core with a Gaussian distribution of vorticity. It implies that our knowledge of the nonlinear effect is restricted to $O(\epsilon)$. For comprehensive lists of theories of vortex rings, the article of Shariff & Leonard (1992) should be referred to.

Motivated by intriguing pattern variation of the dissipation field visualised from numerical data of simulations of fully developed turbulence, Moffatt, Kida & Ohkitani (1994) developed a large-Reynolds-number asymptotic theory for a steady stretched vortex tube subjected to uniform non-axisymmetric irrotational strain. They demonstrated that the higher-order asymptotics satisfactorily account for the fine structure of the dissipation field previously obtained by numerical computation (Kida & Ohkitani 1992). The corresponding planar problem, though unsteady, is dealt with in a similar manner, and an extension of the result of Ting & Tung (1965) to a higher order was achieved by Jiménez, Moffatt & Vasco (1996). The structure of the solutions have much in common; at leading order, a columnar vortex with circular cores, an exact solution of the Navier–Stokes equations, is obtained. A quadrupole component enters at $O(\nu/\Gamma)$, which is realised as the deformation of the core cross-section into an ellipse. The distinguishing feature is that the major axis of the ellipse is aligned at 45° to the principal axis of the external stain. This result leads us to expectat that the strained cross-section of a vortex ring, observed in nature, is established as an equilibrium between self-induced strain and viscous diffusion. Along the line of this scenario, we elucidate the structure of this strained core and its influence on the traveling speed of an axisymmetric vortex ring.

A powerful technique for our purpose is the method of matched asymptotic expansions. It has been previously developed to derive the velocity of a slender curved vortex tube (see, for example, Ting & Klein 1991). However this method is limited to $O(\epsilon^2)$ (Moore & Saffman 1972; Fukumoto & Miyazaki 1991). In the viscous case also, the self-induced strain, with the resulting elliptical deformation of the core, makes its appearance at $O(\epsilon^2)$, and its influences on the translation speed come up at $O(\epsilon^3)$. We are thus requested to extend asymptotic expansions to a higher order.

In §2, we state the general problem. The existing asymptotic formula for the potential flow associated with a circular vortex loop is not sufficient to carry through our program. In order to work out the correct inner limit of the outer solution, we devise, in §3, a technique to produce a systematic asymptotic expression of the Biot–Savart integral accommodating an arbitrary vorticity distribution. In §4, the inner expansions are scrutinised to $O(\epsilon)$ and are extended to $O(\epsilon^2)$. Based on these, we demonstrate in §5.1 that the radii of the loops of the stagnation points, maximum vorticity and vorticity centroid all grow linearly in time owing to the action of viscosity. Thereafter, we establish in §5.2 a general formula for the translating velocity of a vortex ring. In §6, an equation governing the temporal evolution of the axisymmetric vorticity at $O(\epsilon^2)$ is derived, and an integral representation of the exact solution is given, by which the formula of the preceding section can be closed.

A few ambiguous steps lying in previous theories stand as obstacles to proceeding to higher orders. These highlight the significance of the dipoles distributed along the core centerline and oriented in the propagating direction. It turns out that their strength needs to be prescribed at an initial instant, which solves the problem of undetermined constants at $O(\epsilon)$. As a by-product, a clear interpretation is provided of the general mechanism of the self-induced motion of a curved vortex tube. Because of the limitation of space, we must omit the technical details. A comprehensive account of our theory will be available in the paper of Fukumoto & Moffatt (2000).

2 Formulation

Consider an axisymmetric vortex ring of circulation Γ moving in an infinite expanse of viscous fluid with kinematic viscosity ν. We suppose that the circulation Reynolds number Re_Γ is very large:

$$Re_\Gamma = \Gamma/\nu \gg 1 . \tag{2.1}$$

Two length scales are available, namely, measures of the core radius σ and the ring radius R_0. Suppose that their ratio σ/R_0 is very small. We focus attention on the translational motion of a 'quasi-steady' core. This means that we exclude stable or unstable wavy motion and fast core-area waves. Then, according to (1.1), the time-scale under question is of order $R_0/(\Gamma/R_0) = R_0^2/\Gamma$. The core spreads over this time to be of order $\sigma \sim (\nu t)^{1/2} \sim (\nu/\Gamma)^{1/2} R_0$. Our assumption of slenderness requires that the relevant small parameter $\epsilon(\ll 1)$ is

$$\epsilon = \sqrt{\nu/\Gamma} . \tag{2.2}$$

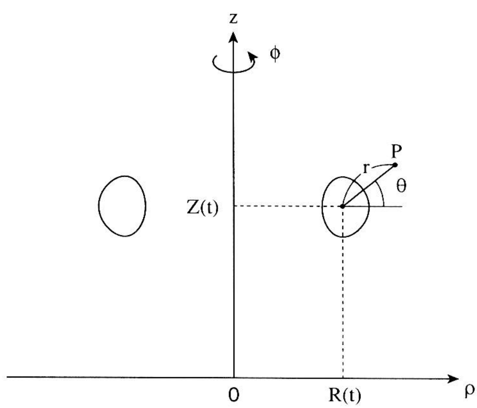

Figure 1

Choose cylindrical coordinates (ρ, ϕ, z) with the z-axis along the axis of symmetry and ϕ along the vortex lines as shown in Figure 1. We consider an axisymmetric distribution of vorticity $\boldsymbol{\omega} = \zeta(\rho, z)\boldsymbol{e}_\phi$ localised about the circle