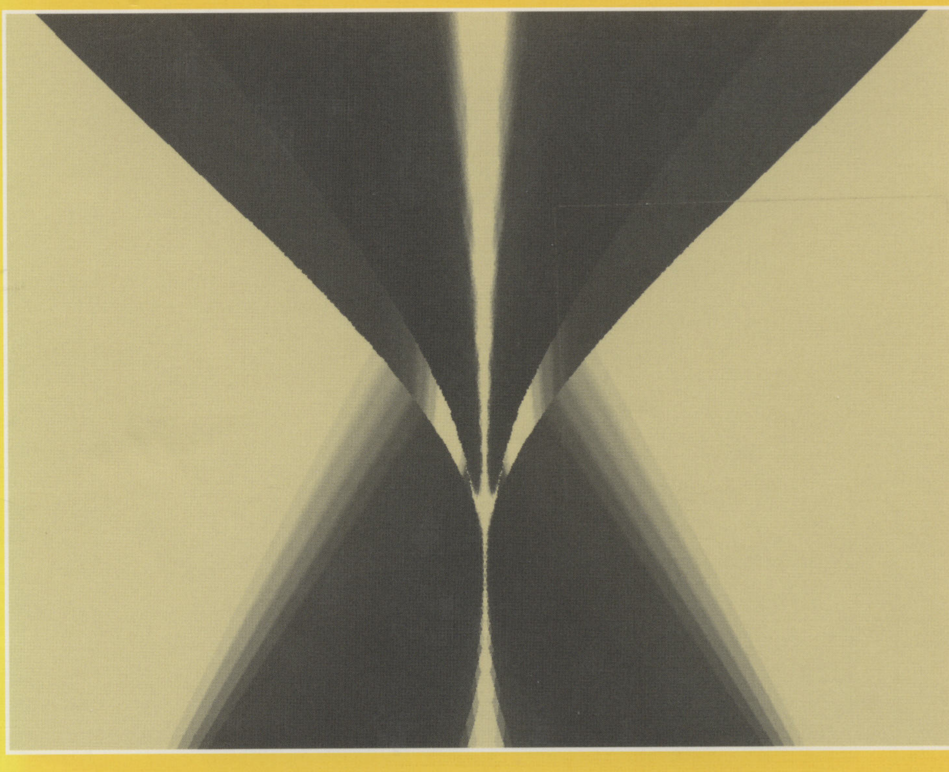
Helge Holden
Nils Henrik Risebro

Front Tracking for Hyperbolic Conservation Laws



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Front Tracking for Hyperbolic Conservation Laws

With 39 Illustrations

Springer

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In memory of Raphael,
who started it all

Preface

Все счастливые семьи похожи друг на друга, каждая несчастливая семья несчастлива по-своему.[1]

Лев Толстой, Анна Каренина (1875)

While it is not strictly speaking true that all linear partial differential equations are the same, the theory that encompasses these equations can be considered well developed (and these are the happy families). Large classes of linear partial differential equations can be studied using linear functional analysis, which was developed in part as a tool to investigate important linear differential equations.

In contrast to the well-understood (and well-studied) classes of linear partial differential equations, each nonlinear equation presents its own particular difficulties. Nevertheless, over the last forty years some rather general classes of nonlinear partial differential equations have been studied and at least partly understood. These include the theory of viscosity solutions for Hamilton–Jacobi equations, the theory of Korteweg–de Vries equations, as well as the theory of hyperbolic conservation laws.

The purpose of this book is to present the modern theory of hyperbolic conservation laws in a largely self-contained manner. In contrast to the modern theory of linear partial differential equations, the mathematician

[1]All happy families resemble one another, but each unhappy family is unhappy in its own way (Leo Tolstoy, *Anna Karenina*).

interested in nonlinear hyperbolic conservation laws does not have to cover a large body of general theory to understand the results. Therefore, to follow the presentation in this book (with some minor exceptions), the reader does not have to be familiar with many complicated function spaces, nor does he or she have to know much theory of linear partial differential equations.

The methods used in this book are almost exclusively constructive, and largely based on the front-tracking construction. We feel that this gives the reader an intuitive feeling for the nonlinear phenomena that are described by conservation laws. In addition, front tracking is a viable numerical tool, and our book is also suitable for practical scientists interested in computations.

We focus on scalar conservation laws in several space dimensions and systems of hyperbolic conservation laws in one space dimension. In the scalar case we first discuss the one-dimensional case before we consider its multidimensional generalization. Multidimensional systems will not be treated. For multidimensional equations we combine front tracking with the method of dimensional splitting. We have included a chapter on standard difference methods that provides a brief introduction to the fundamentals of difference methods for conservation laws.

This book has grown out of courses we have given over some years: full-semester courses at the Norwegian University of Science and Technology and the University of Oslo, as well as shorter courses at Universität Kaiserslautern and S.I.S.S.A., Trieste.

We have taught this material for graduate and advanced undergraduate students. A solid background in real analysis and integration theory is an advantage, but key results concerning compactness and functions of bounded variation are proved in Appendix A.

Our main audience consists of students and researchers interested in analytical properties as well as numerical techniques for hyperbolic conservation laws.

We have benefited from the kind advice and careful proofreading of various versions of this manuscript by several friends and colleagues, among them Petter I. Gustafson, Runar Holdahl, Helge Kristian Jenssen, Kenneth H. Karlsen, Odd Kolbjørnsen, Kjetil Magnus Larsen, Knut-Andreas Lie, Achim Schroll. Special thanks are due to Harald Hanche-Olsen, who has helped us on several occasions with both mathematical and TEX-nical issues. We are also grateful to Trond Iden, from Ordkommisjonen, for helping with technical issues and software for making the figures.

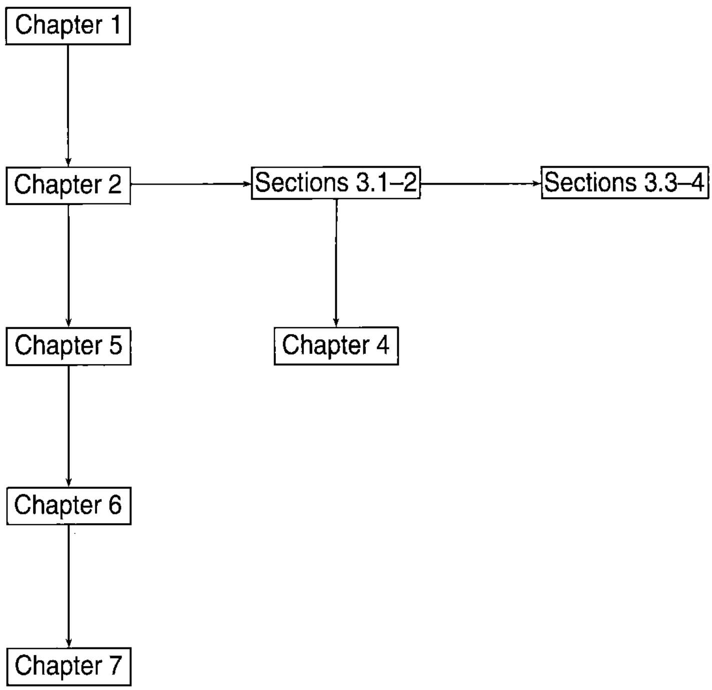
Our research has been supported in part by the BeMatA program of the Research Council of Norway.

A list of corrections can be found at

`www.math.ntnu.no/~holden/FrontBook/`

Whenever you find an error, please send us an email about it.

The logical interdependence of the material in this book is depicted in the diagram below. The main line, Chapters 1, 2, 5–7, has most of the emphasis on the theory for systems of conservation laws in one space dimension. Another possible track is Chapters 1–4, with emphasis on numerical methods and theory for scalar equations in one and several space dimensions.



Contents

1
Introduction

> I have no objection to the use of the term "Burgers' equation" for the nonlinear heat equation (provided it is not written "Burger's equation").
>
> *Letter from Burgers to Batchelor (1968)*

Hyperbolic conservation laws are partial differential equations of the form

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0.$$

If we write $f = (f_1, \dots, f_m)$, $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$, and introduce initial data u_0 at $t = 0$, the Cauchy problem for hyperbolic conservation laws reads

$$\frac{\partial u(x,t)}{\partial t} + \sum_{j=1}^{m} \frac{\partial}{\partial x_j} f_j\left(u(x,t)\right) = 0, \quad u|_{t=0} = u_0. \tag{1.1}$$

In applications t normally denotes the time variable, while x describes the spatial variation in m space dimensions. The unknown function u (as well as each f_j) can be a vector, in which case we say that we have a system of equations, or u and each f_j can be a scalar. This book covers the theory of scalar conservation laws in several space dimensions as well as the theory of systems of hyperbolic conservation laws in one space dimension. In the present chapter we study the one-dimensional scalar case to highlight some of the fundamental issues in the theory of conservation laws.

We use subscripts to denote partial derivatives, i.e., $u_t(x,t) = \partial u(x,t)/\partial t$. Hence we may write (1.1) when $m = 1$ as

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0. \tag{1.2}$$

If we formally integrate equation (1.2) between two points x_1 and x_2, we obtain

$$\int_{x_1}^{x_2} u_t\, dx = -\int_{x_1}^{x_2} f(u)_x\, dx = f\left(u\left(x_1,t\right)\right) - f\left(u\left(x_2,t\right)\right).$$

Assuming that u is sufficiently regular to allow us to take the derivative outside the integral, we get

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\, dx = f\left(u\left(x_1,t\right)\right) - f\left(u\left(x_2,t\right)\right). \tag{1.3}$$

This equation expresses conservation of the quantity measured by u in the sense that the rate of change in the amount of u between x_1 and x_2 is given by the difference in $f(u)$ evaluated at these points.[1] Therefore, it is natural to interpret $f(u)$ as the *flux density* of u. Often, $f(u)$ is referred to as the *flux function.*

As a simple example of a conservation law, consider a one-dimensional medium consisting of noninteracting particles, or material points, identified by their coordinates y along a line. Let $\phi(y,t)$ denote the position of material point y at a time t. The velocity and the acceleration of y at time t are given by $\phi_t(y,t)$ and $\phi_{tt}(y,t)$, respectively. Assume that for each t, $\phi(\,\cdot\,,t)$ is strictly increasing, so that two distinct material points cannot occupy the same location at the same time. Then the function $\phi(\,\cdot\,,t)$ has an inverse $\psi(\,\cdot\,,t)$, so that $y = \psi(\phi(y,t),t)$ for all t. Hence $x = \phi(y,t)$ is equivalent to $y = \psi(x,t)$. Now let u denote the velocity of the material point occupying position x at time t, i.e., $u(x,t) = \phi_t(\psi(x,t),t)$, or equivalently, $u(\phi(y,t),t) = \phi_t(y,t)$. Then the acceleration of material point y at time t is

$$\begin{aligned}\phi_{tt}(y,t) &= u_t(\phi(y,t),t) + u_x(\phi(y,t),t)\phi_t(y,t)\\ &= u_t(x,t) + u_x(x,t)u(x,t).\end{aligned}$$

If the material particles are noninteracting, so that they exert no force on each other, and there is no external force acting on them, then Newton's second law requires the acceleration to be zero, giving

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{1.4}$$

[1] In physics one normally describes conservation of a quantity in integral form, that is, one starts with (1.3). The differential equation (1.2) then follows under additional regularity conditions on u.

The last equation, (1.4), is a conservation law; it expresses that u is conserved with a flux density given by $u^2/2$. This equation is often referred to as the *Burgers equation without viscosity*,[2] and is in some sense the simplest nonlinear conservation law.

Burgers' equation, and indeed any conservation law, is an example of a *quasilinear* equation, meaning that the highest derivatives occur linearly. A general inhomogeneous quasilinear equation for functions of two variables x and t can be written

$$a(x,t,u)u_t + b(x,t,u)u_x = c(x,t,u). \tag{1.5}$$

We may consider the solution as the surface $\{(t,x,u(x,t)) \mid (t,x) \in \mathbb{R}^2\}$ in $\mathbb{R}^3$. Let Γ be a given curve in $\mathbb{R}^3$ (which one may think of as the initial data if t is constant) parameterized by $(t(\eta), x(\eta), z(\eta))$. We want to construct a surface $S \subset \mathbb{R}^3$ parameterized by $(t, x, u(x,t))$ such that $u = u(x,t)$ satisfies (1.5) and $\Gamma \subset S$. To this end we solve the system of ordinary differential equations

$$\frac{\partial t}{\partial \xi} = a, \quad \frac{\partial x}{\partial \xi} = b, \quad \frac{\partial z}{\partial \xi} = c, \tag{1.6}$$

with

$$t(\xi_0, \eta) = t(\eta), \quad x(\xi_0, \eta) = x(\eta), \quad z(\xi_0, \eta) = z(\eta). \tag{1.7}$$

Assume that we can invert the relations $x = x(\xi, \eta)$, $t = t(\xi, \eta)$ and write $\xi = \xi(x,t)$, $\eta = \eta(x,t)$. Then

$$u(x,t) = z(\xi(x,t), \eta(x,t)) \tag{1.8}$$

satisfies both (1.5) and the condition $\Gamma \subset S$. However, there are many ifs in the above construction: The solution may only be local, and we may not be able to invert the solution of the differential equation to express (ξ, η) as functions of (x,t). These problems are intrinsic to equations of this type and will be discussed at length.

Equation (1.6) is called the *characteristic equation*, and its solutions are called *characteristics*. This can sometimes be used to find explicit solutions of conservation laws. In the homogeneous case, that is, when $c = 0$, the solution u is constant along characteristics, namely,

$$\frac{d}{d\xi} u(x(\xi,\eta), t(\xi,\eta)) = u_x x_\xi + u_t t_\xi = u_x b + u_t a = 0. \tag{1.9}$$

◇ **Example 1.1.**

Define the (quasi)linear equation

$$u_t - x u_x = -2u, \quad u(x,0) = x,$$

[2]Henceforth we will adhere to common practice and call it the inviscid Burgers' equation.

with associated characteristic equations

$$\frac{\partial t}{\partial \xi} = 1, \quad \frac{\partial x}{\partial \xi} = -x, \quad \frac{\partial z}{\partial \xi} = -2z.$$

The general solution of the characteristic equations reads

$$t = t_0 + \xi, \quad x = x_0 e^{-\xi}, \quad z = z_0 e^{-2\xi}.$$

Parameterizing the initial data for $\xi = 0$ by $t = 0$, $x = \eta$, and $z = \eta$, we obtain

$$t = \xi, \quad x = \eta e^{-\xi}, \quad z = \eta e^{-2\xi},$$

which can be inverted to yield

$$u = u(x,t) = z(\xi,\eta) = xe^{-t}.$$

◇

◇ **Example 1.2.**

Consider the (quasi)linear equation

$$xu_t - t^2 u_x = 0. \tag{1.10}$$

Its associated characteristic equation is

$$\frac{\partial t}{\partial \xi} = x, \quad \frac{\partial x}{\partial \xi} = -t^2.$$

This has solutions given implicitly by $x^2/2 + t^3/3 = \text{const}$, since after all, $\partial(x^2/2 + t^3/3)/\partial\xi = 0$, so the solution of (1.10) is any function φ of $x^2/2 + t^3/3$, i.e., $u(x,t) = \varphi(x^2/2 + t^3/3)$. For example, if we wish to solve the initial value problem $u(x,0) = \sin|x|$, then $u(x,0) = \varphi(x^2/2) = \sin|x|$. Consequently, $\varphi(x) = \sin\sqrt{2x}$, and the solution u is given by

$$u(x,t) = \sin\sqrt{x^2 + 2t^3/3}, \qquad t \geq 0.$$

◇

◇ **Example 1.3 (Burgers' equation).**

If we apply this technique to Burgers' equation (1.4) with initial data $u(x,0) = u_0(x)$, we get that

$$\frac{\partial t}{\partial \xi} = 1, \qquad \frac{\partial x}{\partial \xi} = z, \quad \text{and} \quad \frac{\partial z}{\partial \xi} = 0$$

with initial conditions $t(0,\eta) = 0$, $x(0,\eta) = \eta$, and $z(0,\eta) = u_0(\eta)$. We cannot solve these equations without knowing more about u_0, but since u (or z) is constant along characteristics, cf. (1.9), we see that