Instructor's Manual for CALCULUS of the elementary functions

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PREFACE

Calculus is to calculate (with a restricted class of functions) and we think this objective should be kept in mind. We believe that the student (whom we often refer to as "S") needs much familiarity with special functions before general discussion of limits (and deeper analysis) has meaning. It seems to us that calculus comes before analysis and we have tried to write a book that would both develop S's power to calculate and bring him to understand some of the processes of analysis via examples. By the end of the course S will be rather good at limits, and the level of sophistication rises as S progresses through the book. Thus one of the reasons for this manual, which we hope will not be thought presumptuous, is to explain to the inexperienced instructor the point of view we had in various places ---and to emphasize what we règard as the essential parts.

Naturally, there is much we could have said in the text but which we left out for fear of obscuring the main argument. So it happens that some of this extra material we have put in this manual. Sometimes what we write in the manual is only to acquaint the instructor with the kind of thoughts that bothered us, or with those subtleties that we wished to avoid. In other places this extra material may be found interesting to use in class.

Thus the comments in this manual are both of a pedagogical and a mathematical nature. We apologize to the instructor if our remarks seem at times gratuitous or patronizing. It is not our intent to be so. The manual is meant to be helpful, not irritating. A rather complete solutions manual is also available.

Part I

DIFFERENTIAL CALCULUS

In this Part, Chapters 1, ... , 8, we are concerned with derivatives of functions of one variable. Our principal objectives are (a) to develop skill in finding derivatives and (b) to acquaint S with the simplest applications, and so give him a feeling of power. There is little padding and all chapters should be taken up at least in part. We regard everything as necessary with the exception of Section 4, Chapter 3, Sections 7, 8, 9, Chapter 6, and Section 4, Chapter 8.

There is an abundance of problems and we feel that at least half of them, in this part, should be done by all students.

Chapter I

THE DERIVATIVE

I The Problem of Tangents.

We do not fuss with a formal definition of limit nor do we prove theorems about limits that seem obvious to students. It is our experience that the simple limits we need are readily handled by inspection. For example $\lim_{x \to 2} (x^2 - x)$ presents no problem. To give an $\varepsilon - \delta$ proof that this limit is 2 strikes most students as "busy work." All that is needed is some attention to notation.

This is not the time to refer to Appendix C. Later on, good students can benefit from reading it over, but it is not (in our opinion) part of the course. It should not intrude upon class time.

Discussion of the figures on page 4 can be used to get students to generate the definition of the tangent line. It may be necessary to remind students about inclination of lines. There will be a tangent line if the inclination of the secant line has a limit. Then vertical tangents are possible, though they have no slope.

Students need to see that Δx can be positive or negative.

The examples and problems are standard. It is desirable that S sketch the graphs of the functions. Then inspection of the graph and comparison with the slope m of the tangent is fruitful. It is essential that S compute m from the definition. If S has had some calculus he will want to write the slope without applying the definition.

2 The Derivative.

This is standard material. S is to compute the derivative directly from the definition. As in Section 1 it is beneficial to draw pictures with $\Delta x < 0$ as well as $\Delta x > 0$.

Some attention should be given to the word "operator." The derivative operator is a function whose domain is the set of differentiable functions. It maps differentiable functions into functions. The graphical construction of f' from f on page 10 affords the first chance to observe (without proof) that if $f'(x) > 0$ on an interval then f is increasing in that interval.

Problems 1, ... , 9 are worth doing in class, or having S present. Problem 13(d), rather surprisingly, gives students trouble. Problem 23 deserves attention in class, if assigned. For $x \neq 0$ the derivative is easily obtained treating the cases $x > 0$ and $x < 0$ separately. S should try to compute the derivative at $x = 0$. The difference quotient is

$$\frac{f(0 + \Delta x) - f(0)}{\Delta x} = \frac{|\Delta x|}{\Delta x}$$

and the limit does not exist.

3 The Fundamental Lemma.

Much use is made of the Fundamental Lemma (FL). Not only is it used here and in Chapter 2, but its analogue is essential for functions of several variables. See Chapter 14.

In the FL η is a function of two variables, x and Δx; this needs to be mentioned in class but notation showing this has been avoided as we find it leads to confusion. That is, to write $\eta(x,\Delta x)$ seems to bother S. The students should compute η only in enough cases to see what it is (Problems 10,...,14). Computation of η can lend meaning to the calculation of derivatives.

Continuity needs discussion in class. Then the Corollary on page 14 discussed. Observe that there is no ε - δ proof here. It would be inappropriate at this stage. All that is needed is to observe that by the FL

$$f(\overline{x}) - f(x) = f(x + \Delta x) - f(x) = f'(x)\Delta x + \eta\Delta x$$

Then one simply remarks that as $\overline{x} \to x$ $f(\overline{x}) \to f(x)$. One must emphasize that the converse of the Corollary is false as the two figures at the bottom right of page 14 suggest.

The various notations for the derivative are learned while doing the problems. Only the definition is to be used. Enough should be assigned to begin the development of algebraic skills.

4 Rates; Velocity; Rate of Change.

This is standard material. The main problems are linguistic:

(1) Average rate of change needs illustration, e.g., via automobiles.

(2) S needs to be coerced into accepting that instantaneous rate of change is a definition.

Observe that here we use the Theorem on page 18 without proof. Students accept this without a quiver.

In Problems 11, ... , 16 S should follow the moving point on the line. Problem 18 can be used to comment on the various motions possible that conform to this same set of discrete data.

5 Antiderivatives.

This brief section is adequate for much of the early needs in physics courses. At this stage S knows that $Dt = 1$, $Dt^2 = 2t$, $Dt^3 = 3t^2$ and perhaps a few other cases. There is no attempt here to develop technique. After Chapter 2, when S has some skill in finding derivatives, he will have no problems in finding the antiderivatives he needs for elementary physics.

Do not (we think) use the term "indefinite integral" now.

Two things are important in this section: (1) The definition of antiderivative. (2) The theorem on page 21.

In Example 1, S is asked to guess an antiderivative. There is no rule to follow. Example 2 provides a rule.

Example 4 merits particular attention. If S can see that f' can predict f (by following one's nose, so to speak) then S has grasped an important method for the solution of differential equations. Problems 1, ... , 6 are similar and usually easy for S.

Mathematical and Historical References.

Students of mathematics should read about mathematics and mathematicians as well as textbooks of mathematics. Bell in particular can stimulate interest in even weak students. Apostle is only for the better student who wants proofs. Boyer is probably a bit erudite for most at this stage, but the pages referred to are quite readable and relevant. Cajori, Eves, Struik are but three of many histories available.

Chapter 2

THE TECHNIQUE OF DIFFERENTIATION

Three remarks may suffice to convey our approach.

1. The chapter is designed to develop technique---and this means algebraic facility as well as facility in the use of the formulas. About half of the problems should be done by all students and weaker students perhaps should do more. Sometimes the real meat of a problem is in the algebraic simplification _after_ differentiating. Answers to most problems are given and part of the game is to get the answer in the form given.

2. The complete list of differentiation formulas is developed. We see no point in deferring the derivatives of the transcendental functions until later. In fact, we think that harmful.

3. The chain rule, formula XX, occurs late and is not really needed until much later. All the basic formulas are for derivatives of composite functions.

1 The Differentiation Formulas.

The entire list of differentiation formulas is collected here for easy reference. Note that we use the Leibnizian, $\frac{d}{dx}$, notation as it seems, at this stage, to be more transparent. The problems use all notations.

The problems require, almost exclusively, only formulas I, ... , IV, and the special case of VI. Though the problems are trivial, effort should be made to get S to write the answer in "simplified form" in as few steps as possible.

Problems 24, ... , 28 form a sequence. In 25, 26 the general power formula VI can be avoided. Problems 29, ... , 31 require VII unless negative exponents are used in which case VI is needed. In Problem 42 we found that students often didn't know what a polynomial was and also did not like to call constant functions polynomials.

2 Derivatives of Products, Powers, and Quotients.

The derivation of formula V occurs as Problem 60 and we think should be assigned. S can follow the pattern of the proof of VII.

The proof of VI is for the case where n is a positive integer. However, we use VI for n arbitrary. A proof for arbitrary n occurs in Section 4. It is a good problem to assign to good students a proof of VI for n a rational number. This to follow assignment of Problem 62.

In some problems algebraic simplification before differentiating can result in easier work, for example in Problems 6, 7, 8, 12, 19, 37, 56. Problem 10 follows from Problem 9 by replacing a by -a.

Problems 35, 36 are paired. See also 58.

Problems 53, 54, 55 are related.

Problem 57 can be used for oral work in class. S should practice economical exposition.

In Problem 63 one gets the derivatives automatically if one uses $|u| = \sqrt{u^2}$. Thus, in (b),

$$\frac{d}{dx}|x^2-9| = \frac{d}{dx}[(x^2-9)^2]^{\frac{1}{2}} = \frac{1}{2}[(x^2-9)^2]^{-\frac{1}{2}}(2)(x^2-9)(2x) = \frac{2x(x^2-9)}{|x^2-9|}$$

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3 Derivatives of Logarithmic Functions.

We assume that S is familiar with exponential and logarithmic functions. Reference to Appendix A, page 498 may be helpful.

There are serious difficulties associated with a rigorous treatment of these functions, difficulties which are usually glossed over in pre-calculus mathematics, and which are not mentioned here either. The steps and difficulties are as follows.

1. a^x for x an integer is defined as in elementary algebra and the laws of exponents then proved by induction.

2. a^x for x rational, $x = p/q$ with p,q integers and $q > 0$, is defined by $a^x = (\sqrt[q]{a})^p$. The laws of exponents for rational exponents are then easily established, as is the increasing character of a^x for $a > 1$.

3. a^x for irrational x is defined as the limit of a^r, for r rational, as $r \to x$.

Now there are many non-trivial things to prove: (a) The limit in (3) exists. (b) The laws of exponents. (c) The continuity of the exponential function. Each of these requires a careful $\varepsilon - \delta$ proof. These difficulties are to be avoided at this level. More relevant for teaching is the fact that the student does not see the problems. He has an intuition how a^x behaves and can sketch its graph.

Once the exponential function, $\exp_a$, has been defined and its properties noted, the properties of its inverse, $\log_a$, are easy to establish.

The development in the text presumes the above background, which is sketched in Appendix A. It is our experience that students are already familiar enough with these functions and are ready to use their properties in finding their derivatives.

There is another approach to exponential and logarithmic functions that has been used in some calculus texts. This alternative approach avoids the difficulties (a), (b), and (c) above by defining the logarithm function as an integral and then obtaining the properties of this function from properties of the definite integral. Then the exponential function is defined as the inverse of the logarithm function. It is our belief that this sophisticated approach is too elegant for most students and is hard to motivate, and besides some of the proofs are not easy. But more important, it seems to us, is that with this approach one cannot finish the basic differentiation formulas until after the definite integral has been studied. It is enough for S, now, to master the concept of differentiation without being distracted by the more difficult concept of the definite integral. In brief outline the steps in this alternative approach are as follows:

(i) Define a function L by

$$L(x) = \int_1^x \frac{dt}{t}, \quad \text{for } x > 0$$

Then $\frac{d}{dx} L(x) = \frac{1}{x}$, so L is increasing and continuous.

(ii) Prove the "laws of logarithms" for L.

(iii) Prove that the range of L is the entire real line.

(iv) Define E as the inverse of L, $E = L^{-1}$.

(v) Define e by L(e) = 1. Then prove $E(n) = e^n$ for n an integer, and further that $E(r) = e^r$ if r is rational.

There are still a few matters to be cleared up but the essentials are now done.

In the text derivation of VII which uses the Fundamental Lemma the chain rule, XX, has, in effect, been used without specific mention. We think this helps S to understand the chain rule when it occurs. The proof is otherwise quite standard. Very good students may benefit from perusing Section 1 of Appendix D at this time.

Though the emphasis is on natural logarithms, S may feel better if he sees their relation to common logarithms: $\log N = (\log 10) \log_{10} N \approx 2.303 \log_{10} N$.

The laws of logarithms are useful in Problems 4, 5, 6, 7, 8, 10, 13, 16, 18, 22, 24, 30, 33, 34, 35. Note the different domains in 9 and 10.

Problem 32 will be important later for antiderivatives.

4 Exponential Functions; Logarithmic Differentiation.

Students will need a word of explanation about the notation $\underline{\exp_a}$. It may suffice to say that we have invented a special name for the function given by a^x, just as we have a special name for the trigonometric function $\underline{\sin}$. The inverse of $\underline{\exp_a}$ then is denoted by $\underline{\log_a}$, so $\underline{\exp_a}^{-1} = \log_a$. The special case, $\underline{\exp_e}$, is denoted simply by $\underline{\exp}$ and is called the exponential function. If S is confused by the notation it won't matter as we use this notation very little and introduce it only because S may encounter it later.

It is perhaps worth noting that

$$\exp_a \circ \log_a = \text{identity function on the positive reals}$$

or

$$a^{\log_a x} = x \quad \text{for } x > 0$$

$$\log_a \circ \exp_a = \text{identity function on all reals}$$

or

$$\log a^x = x \quad \text{for } -\infty < x < \infty$$

The derivation of formula IX is handled via the chain rule rather than formula XXI for inverse functions. The same method is used for the inverse trigonometric functions and could be used for deriving XXI.

Example 2 is to convince S that De^u is adequate for all cases. There are very few problems Da^u with $a \neq e$ and these need not be assigned.

Example 3 establishes, finally, the general power formula, VI, and should be discussed in class.

The new formula of Example 4 will be used by some S in spite of urging them to use logarithmic differentiation.

Example 6 is relevant for Problems 41, ... , 46.

The problems need little comment. Problems 29 and 30 should be assigned together if at all, and then also Problem 47.

5 Derivatives of the Trigonometric Functions.

The graphical "derivation" of D sin = cos helps to convince S that drawing pictures helps.

The derivation of D sin = cos is standard. Observe that the continuity of the sine and the cosine at 0 has been used when we evaluate the limit of (4). These are presumed well-known from trigonometry and the point is not belabored in the text.

Formulas XII, ... , XV should be derived by S.

The problems are mostly traditional. There are numerous opportunities to use trigonometric identities to get the answer given in the text, and the problems are constructed to promote this facility as well as skill in differentiating.

In some problems attention should be given to the domain of the function, e.g., in 31 and in 49 as compared with 50.

In Problem 63 it is best if S uses a table with the argument given in degrees, for that will tend to emphasize the convenience of radian measure.

Problems 66, 67 can cause confusion. What is intended, for example, in 66, was for S to express tan θ in terms of sin θ: $\tan\theta = \frac{\sin\theta}{\cos\theta} = \sin\theta/\sqrt{1-\sin^2\theta} = u/\sqrt{1-u^2}$ where $u = \sin\theta$. Then S computes $\frac{d}{du}\frac{u}{\sqrt{1-u^2}}$. We have had students who have had (unfortunately) some prior calculus do the following:

$$d\tan\theta = \sec^2\theta\, d\theta\ ,\quad d\sin\theta = \cos\theta\, d\theta$$

whence

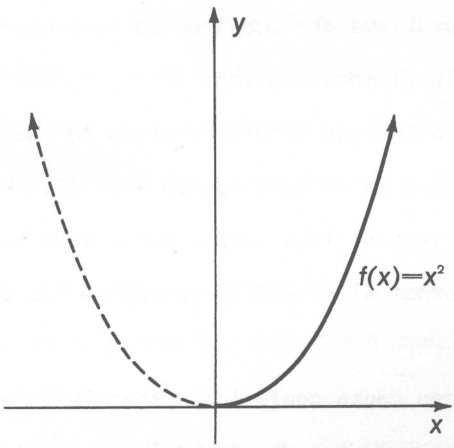
$$\frac{d\tan\theta}{d\sin\theta} = \sec^3\theta$$

Naturally, this is out of place at this stage, although correct.

6 The Inverse Trigonometric Functions.

Inverse functions were encountered with $\exp_a$ and $\log_a$ but now it seems more natural to emphasize the "inverse function language." S should see that without restricting the domains of sine, cosine, tangent, and cotangent their inverse relations arcsin, arccos, arctan, arccot are not functions (i.e., single valued). At this time it is helpful to recall the example of the inverse to

$$y = f(x) = x^2$$



To get a unique inverse one restricts the domain of f to the non-negative reals, $x \geq 0$. Then the unique inverse is the square root function: $x = \sqrt{y}$, the principal value for the square root of y. Note that we have no symbolic name for this function. We do not, as might be natural, speak of the function "$\sqrt{\ }$."

The choices for the principal values of the inverse sine, cosine, tangent, and cotangent are natural. It turns out that any choice for the inverse secant or cosecant is inconvenient. Conse-