Introduction to Discrete Linear Controls

Theory and Application

Albert B. Bishop

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Preface

This book is an introduction to discrete linear controls. It is written for those engineers, operations researchers, and systems analysts involved with the design, analysis, and operation of discrete-time decision processes. The basic theory is developed directly from the underlying discrete mathematics in an effort to provide the user with an understanding of the nature of discrete controls and equip him in as simple and straightforward a manner as possible with the necessary tools and techniques to deal with such systems.

This approach is somewhat rare in the current control theory literature, in which the theory of discrete controls is developed as an extension of the theory of continuous systems and usually in the context of electromechanical circuits.† For those whose interests lie in the areas of conceptual models of various discrete man–machine systems or the automation of inherently discrete production processes, the presentation here precludes the necessity of devoting time and energy to the learning of classical continuous controls in order to eventually gain the material they need. This is not to say that the continuous theory learned might not be useful, especially if the processes involved do have continuous features in their operation, but for many purposes discrete-systems theory will fully suffice. Furthermore, as computer control of manufacturing processes continues to advance and as quantitative analysis and optimal design of an ever-widening variety of societal and ecological systems involving human decision makers emerges, the appropriateness of discrete models and hence the need for ready access to the methodology of discrete control systems is rapidly increasing.

As in most subjects, the more extensive the background, both analytical and empirical, that the reader brings to his study of the material herein, the greater his potential for not only gaining a basic understanding of the material presented but also extending it in innovative ways. Throughout the study of the text, the reader is encouraged to question and explore, to develop more realistic or effective models, and to devise new approaches to the derivation and manipulation of models and the simplification of calculation procedures. Above all, he should be continually searching his field of experience for areas of application. It has been assumed in preparing the text, however, that the

† See, for example, *Digital and Sampled Data Control Systems* by J. T. Tou (1959, see reference list for detailed information) or *Discrete-Time and Computer Control Systems* by J. A. Cadzow and H. R. Martens (1970). The latter develops discrete control theory with a minimum of dependence on continuous system theory. A recent book devoted completely to discrete theory, and hence an exception to the point being made here is *Discrete-Time Systems* by J. A. Cadzow (1973).

reader will have a background in both differential and integral calculus and be familiar with the basic concepts of classical optimization theory for analytical functions. Although numerous opportunities exist for applying a variety of alternative optimization techniques, these are either mentioned in passing or left entirely to the reader. Sufficient knowledge of probability theory for the reader to be familiar with basic definitions, notions of independence, moments, joint moments, and common distributions is assumed, although much of this material is reviewed briefly in the context of its usage in the text. On the other hand, no particular knowledge of discrete mathematics is assumed. The calculus of finite differences and solution procedures for linear difference equations with constant coefficients is covered in detail in Chapters III–VI. In addition, only a cursory familiarity with the notions of limits is assumed. This is in spite of the fact that the applicability of the z transform depends on the convergence of an infinite sum, the conditions for which we state and then assume hold from there on. No background whatsoever in control theory is assumed.

The book provides a series of building blocks upon which one can formulate models and devise analysis and design exercises which can extend the coverage in the text to best suit the background and interest of the teacher and students. Specifically, Chapter I is a basic introduction to systems analysis, discrete systems, the concept of control, and the role of models in system analysis and design. In Chapter II the development of system difference equations is illustrated with respect to a generalized discrete-process control system, a production–inventory control system, and a simplified flow analysis of the criminal justice system. Chapter III introduces some concepts from the calculus of finite differences useful in the formulation and solution of difference equations. Solution of linear difference equations with constant coefficients by classical means is discussed in Chapter IV. Chapter V introduces the z transform as a more flexible approach to the formulation and solution of linear difference equations, and Chapter VI presents the inverse transformation. In Chapter VII criteria for evaluating system performance are discussed. This is followed by examining the performance of a simplified first-order process control system when perturbed by each of several types of common system disturbances. This performance evaluation is extended in Chapter VIII to include the effects of measurement and sampling errors and a series of examples is presented to illustrate the selection of an optimal value for the control system parameter for each of several types of disturbance given several possible performance criteria. Chapter IX is devoted entirely to system stability and tests to determine the conditions under which a system will operate stably. The properties and performance of several types of second-order system are presented in Chapter X. Emphasis is given to the analysis of the ranges of parameters for stable operation and the interrelationships between these parameter values and the effects of random measurement errors. Chapter XI considers extensions to higher order systems. The signal-flow graph is introduced here as a convenient means of representing and manipulating complex systems. Effects of delay in sensing and feeding back information for decision-making purposes is included among several miscellaneous concluding topics.

The exercises at the end of each chapter are designed to extend the material presented in the text. Each chapter has several drill-type problems to test understanding of each new topic and technique from that chapter. Many of these are presented sequentially so that a course instructor will always have some problems he can assign upon completion of each section or subsection. As new steps in problem solution are covered in the text, exercises are available to apply that step to the results of previously completed steps. Other exercises provide opportunity for additional study of systems or techniques or require verification of expressions presented in the text with only partial or no derivation. Others force attention to new formulations or areas of application. A few might be considered minor topics for research. Because of the building-block nature of these exercises, there are numerous cross references among them. Difference equation models developed in exercises in Chapter II are solved in several stages by classical means in exercises in Chapter IV and by the z transform in Chapters V and VI. System performance under a variety of environmental conditions is evaluated in exercises in Chapters VII, VIII, or X and stability established in exercises in Chapter IX. It is hoped that the familiarity the reader gains with a few specific systems in this way will permit concentration on each new topic as it is introduced without having to feel out a new system structure at every turn.

Acknowledgments

The author is indebted to many people, only a few of whom can be acknowledged here. I was first attracted to the industrial engineering–operations research arena because of an intense fascination with the results one of my instructors could get from mathematical representations of bits and pieces of quality control and production control systems. This man, who later became my graduate adviser, friend, and continuing source of stimulating ideas and encouragement, was Loring G. Mitten, now chairman of the Management Science Division at the University of British Columbia. A powerful member of his profession and the world's best and most unselfish adviser, the true extent of his contributions can be fully understood only by his advisees. It was he who suggested a text of this kind almost twenty years ago, and whose continued encouragement led, at long last, to its completion. Out of respect and appreciation, I dedicate this book to him. I hope it is worthy of his high standards.

I also acknowledge the help provided by David Baker and William Morris, my department chairmen during the lengthy writing process, in making available the resources of The Ohic State University Department of Industrial and Systems Engineering to assist in this undertaking. Numerous students commented on and weeded out errors in several versions of course notes. All such efforts are appreciated, but Seetharama Narasimhan deserves special mention for his assistance with both text and exercises. Salah Elmaghraby performed an outstanding comprehensive review of an early manuscript which led to numerous substansive changes, all of which should result in significant improvements. The patient typists who readied the material for course notes and text manuscript deserve special credit for enduring this impatient author. My thanks to Mrs. Cindy Dickinson, Mrs. Judy Crowl, Mrs. Lois Graber, Mrs. Carol McDonald, and especially to Miss Joan Case who single-handedly typed the entire final manuscript. Finally, my deepest thanks to my wife, Louise, and my children, John, Sue, and Jim, for their love and support which were so essential to bringing this task to completion.

Contents

Chapter I Systems Theory and Discrete Linear Control Systems

The discrete, linear, time-invariant system has in recent years become an object of increasing interest in many areas. Much of this attention has come from operations researchers and systems analysts involved with either human decision makers who tend to make specific, individually identifiable decisions or digital or time-shared system components whose information outputs occur periodically. Furthermore, because of the relative simplicity of mathematical models of such systems, beneficial insights can often be gained by modeling a wide variety of systems as though they were discrete, linear, and time invariant.

In this first chapter we introduce the concept of a system and then define the basic terms of discreteness, etc., in the context of a system. The notion of control and the components and structure of a control system are then examined. The chapter closes with a discussion of models and their role in system analysis and design.

1.1 Systems Theory

The 1950s saw the rise of "operations research" with its emphasis on finding optimal solutions to operational problems (Churchman *et al.*, 1957). An oft-stated feature of OR methodology is a "systems approach," which basically means that the researcher should carefully strive to consider all those factors which are likely to have a reasonably significant effect on the solution of the problem. For example, the routing and scheduling of trucks among terminals of a common carrier cannot be done properly without consideration of the company's truck maintenance program and the materials handling capability at the loading docks. Aircraft instruments and controls have to be designed with both the motor skills and the information processing capacity of the pilot explicitly in mind. Of particular importance to the industrial engineer, the layout of single work stations has had to give way to production line design involving not only production but also materials handling and storage.

With the systems approach came the compiling of lists of "pertinent" factors as one of the first steps in any problem-solving effort. To staff a tool crib one needs to know the types and numbers of tools handled, the frequency of requests for each type by time of day and week, and the service-time distributions. An added frill might be the interference patterns that result from the presence of more than one attendant. Further study usually produces additions and deletions from the list and some understanding of the interrelationships among the items listed. This identification of items to include in the system and the descriptions of their interrelationships is termed *model building*. The description itself is the *model*, which serves both as a source of learning and insight and as a vehicle to optimize system structure and performance.

In order to obtain or even define an optimum solution to a problem, the problem-solver needs a criterion of optimality and a scale on which to evaluate competing solutions. In industrial settings one usually seeks to maximize profit or to minimize cost, although surrogate measures involving product quality, adherence to deadlines, and customer service, all of which contribute to profit in complex ways, are often used. The trucking company may attempt to minimize delivery time or damage to freight. The industrial engineer could attempt to maximize throughput or minimize the bank sizes of his production line. Elsewhere, particularly in the public sector, benefit or effectiveness often share the spotlight or even replace profit and cost as the basis of evaluation. Both cost and performance must be explicitly considered by the designer of an interceptor missile system, where performance could involve maximizing the probability of intercept or minimizing the damage inflicted by an attack force. The aircraft cockpit designer, however, is essentially completely interested in flight safety with equipment cost involved only as a constraint, if at all.

The systems approach of the operations researcher has undergone considerable extension and formalization in recent years resulting in what many refer to today as *systems theory*. The history of this evolution and discussions of the principal current formulations are presented by Klir (1972). Brockett (1970) presents an engineering oriented discussion of linear systems, and Howard provides extensive coverage of dynamic probabilistic systems in his two-volume set divided into Markov (1971a) and semi-Markov and decision processes (1971b). In this book we will be extensively involved with the systems approach of model building, criterion formulation, and optimization of performance of discrete linear decision systems, referred to here as *control systems*. This is the type of system of particular interest to the manager, public official, operations researcher, and design engineer. We will draw heavily on available systems theory, but only to the extent necessary to motivate, derive, and explain the points being developed. The reader is referred to the sources listed above for further discussion of systems theory.

Because of our primary interest in discrete systems, a discussion of what is meant by "discrete," "discrete system," and other terms basic to our exposition is in order at this point. We will then turn our attention to control theory and introduce the concept of a decision or control system. The chapter concludes with a general discussion of models and their role in systems analysis and design.

1.2 Discrete Systems

A discrete event is a specific happening readily distinguishable from other events. Examples include the inauguration of a president, the opening of a supermarket, the dispatching of a bus, or the completion of the manufacture of the ith engine block in a production run. Often, however, the discrete character of an event is a matter of definition. For example, the flow of water through a hydroelectric station is, under normal operating conditions, a continuous phenomenon. Yet one could define as a discrete event the passing of the one-billionth gallon through the station. Similarly, the height of water in a reservoir is a continuous variable. Yet it can be discretized by measuring to the nearest foot only and attaching an integer (discrete) measure to the level. Time is often described in discrete terms such as the number of days to repay a loan. It may also frequently be expressed in units corresponding to the occurrence of a sequence of discrete events. For example, *time* i could be defined as the time at which the ith engine block is completed or as the end of the ith week in a production control plan in which factory schedules are issued weekly. Obviously, to convert to clock or calendar time, i must be multiplied by the time between events and the result added to the time corresponding to the origin of the sequence.

As used herein a *discrete system* is one whose output occurs naturally on a discretized time scale, often referred to as *discrete time*. The engine-block manufacturing line is a good example. It is obviously not meant that the line exists or operates only at those instances at which a block is finished, but that the meaningful descriptors of the operation of the line are, for the most part, the characteristics of the block produced. Since each succeeding set of such characteristics is attached to succeeding engine blocks, it can also be ascribed to the discrete points in time at which the blocks are completed. When described in this way, the engine-block line is a discrete system.

The discreteness of the process output, however, is not the only factor which determines the discreteness of a system. A Fourdrinier machine produces paper in a continuous sheet. The quality characteristics such as density and moisture content are determined, however, by moving a gage across the bed of the machine. At the completion of a scan, the gage signals are analyzed and a discrete-control action initiated to adjust for any noted deviations from standard. Thus the control of this continuous product is accomplished by a discrete-control system. Similarly, a central computer which sequentially monitors a number of processes on a time-shared basis supplies each unit in turn with a discrete-control signal regardless of the nature of the processes or their outputs.

In summary, the term discrete system refers in this book to any system whose operation or output is conveniently described on a discrete time scale; although, in general, the system characteristics, such as the height of an individual engine block, are given continuous measures. Many authors prefer the term "discrete-time system," which is really a more apt description of what is meant. In general, the index i is used to refer to discrete time. As stated previously, i is related to continuous time t by the relationship

$$i = t/T, \qquad i \text{ integer}, \tag{1.2.1}$$

where T is the time between events. Usually, functions of discrete time are written simply in terms of the argument i, e.g., $f(i)$,with T suppressed. However, where real-time considerations are important, conversion from $f(i)$ to $g(t)$, the comparable function in real time, is accomplished simply by substitution of t/T for i in $f(i)$. For example, the function of discrete time

$$f(i) = 3i^2 + 2i$$

can be expressed in terms of continuous time t as

$$g(t) = \frac{3}{T^2}t^2 + \frac{2}{T}t.$$

For $T = 2$,

$$g(t) = 0.75t^2 + t.$$

Conversely, for

$$g(t) = t^3 + 3t$$

and $T = 2$,

$$f(i) = 8i^3 + 6i.$$

Two additional properties which will usually be assumed for the systems discussed herein will now be defined. These are the properties of "linearity" and "time invariance." The reader is referred to the work of Howard (1971a, Chapter 2) for a complete and well presented treatment of the theory of discrete, linear, time-invariant systems.

Linearity

As is well known to engineers, analysts, and operations researchers, a linear relationship between a dependent variable and a group of independent variables is one which can be expressed as a linear surface, i.e., as a line, plane, or hyperplane. If it requires 1.5 minutes to test a circuit board regardless of whether it is the first, seventeenth, or whichever number tested during a production run, the time t required for the test can be expressed as $t = \sum_{k=1}^{n} 1.5 = 1.5n$, where n is the number of circuit boards in the current batch. This equation is, of course, the equation for a straight line passing through the origin, and we say that the total test time is a linear function of the number of items to be tested. Similarly, if the direct cost to manufacture one unit of product of type k is c_k, regardless of how many items of that type have already been produced and what types and how many of other kinds of items are being made, total production cost C can be expressed as

$$C = c_0 + \sum_k c_k n_k, \tag{1.2.2}$$

where n_k is the number of units of product type k manufactured and c_0 represents fixed costs. Equation (1.2.2) is the equation of a hyperplane, a linear surface, with cost–axis intercept c_0.

To extend the notion of linearity to discrete systems, consider Fig. 1.2.1.

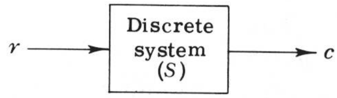

FIG. 1.2.1. Discrete system with input r and output c.

r and c are vectors of discrete-time inputs and outputs, respectively, and are often referred to as input and output *signals*. S represents the transformation performed by the discrete system on the input to produce the output. It is often referred to as the *system operator*. Specifically, using i as the index of discrete time,

$$r = \{r(0), r(1), \ldots, r(i), \ldots\},$$

$$c = \{c(0), c(1), \ldots, c(i), \ldots\},$$

and

$$c = S(r).$$