Lecture Notes in Statistics

142

György Terdik

Bilinear Stochastic Models and Related Problems of Nonlinear Time Series Analysis

A Frequency Domain Approach

Springer

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Library of Congress Cataloging-in-Publication Data

Terdik, György.
Bilinear stochastic models and related problems of nonlinear time series analysis : a frequency domain approach / György Terdik.
p. cm. -- (Lecture notes in statistics ; 142)
Includes bibliographical references and index.
ISBN 0-387-98872-6 (softcover)
1. Time-series analysis. 2. Nonlinear theories. I. Title.
II. Series: Lecture notes in statistics (Springer-Verlag) ; v. 142.
QA280.T44 1999
519.5'5--dc21 99-23873

Printed on acid-free paper.

Camera-ready copy provided by the author.
Printed and bound by Sheridan Books, Ann Arbor, MI.
Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-988726 Springer-Verlag New York Berlin Heidelberg SPIN 10729680

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Introduction

"Ninety percent of inspiration is perspiration."

[31]

The Wiener approach to nonlinear stochastic systems [146] permits the representation of single-valued systems with memory for which a small perturbation of the input produces a small perturbation of the output. The Wiener functional series representation contains many transfer functions to describe entirely the input-output connections. Although, theoretically, these representations are elegant, in practice it is not feasible to estimate all the finite-order transfer functions (or the kernels) from a finite sample. One of the most important classes of stochastic systems, especially from a statistical point of view, is the case when all the transfer functions are determined by finitely many parameters. Therefore, one has to seek a finite-parameter nonlinear model which can adequately represent nonlinearity in a series. Among the special classes of nonlinear models that have been studied are the bilinear processes, which have found applications both in econometrics and control theory; see, for example, Granger and Andersen [43] and Ruberti, et al. [4]. These bilinear processes are defined to be linear in both input and output only, when either the input or output are fixed. The bilinear model was introduced by Granger and Andersen [43] and Subba Rao [118], [119]. Terdik [126] gave the solution of a lower triangular bilinear model in terms of multiple Wiener–Itô integrals and gave a sufficient condition for the second order stationarity. An important representation in terms of the generalized transfer functions was given by Priestley [98] and in terms of higher order spectra by Brillinger [17].

The present work is intended to be a systematic statistical analysis of bilinear processes in the frequency domain. The first two chapters are devoted to the basic theory of nonlinear functions of stationary Gaussian processes; Hermite polynomials; cumulants; higher order spectra; multiple Wiener–Itô integrals; and finally, chaotic Wiener–Itô spectral representation of subordinated processes. Chapter 3 contains the results concerning bilinear processes. For an easier understanding of the technique of chaotic representation, three levels of bilinear processes are considered: the simple bilinear model, the general bilinear model with scalar value, and the multiple bilinear model. In each case explicit assumptions of second order stationarity and expression for the second order spectrum are given. The assumptions of the existence of the $2n$th order moments are proved for the general bilinear model with scalar value, and an expression for the bispectrum is obtained. The Generalized Autoregressive Conditionally Heteroscedastic (GARCH(1,1)) model is investigated by the same methods as the bilinear one, and its basic spectral properties are shown. The bilinear realization for Hermite degree-N homogeneous polynomial model and its minimal realization are also considered. There are two chapters for general nonlinear time series problems. Chapter 4 covers the non-Gaussian estimation. It was Brillinger [21] who suggested using for parameter estimation not only the spectrum but the bispectrum as well. We give explicit expression for the asymptotic variance of this estimator and prove the asymptotic normality and consistency. The asymptotic variance in the case of linear non-Gaussian processes is expressed in terms of skewness and kurtosis. This method is used for the parameter estimation of bilinear processes. The other general problem is in Chapter 5, where we consider the linearity of a time series. We use a weak notion of linearity of a time series and give a bispectrum-based test for checking it.

Further references and historical comments on the frequency domain approach to the time series analysis and nonlinear models are provided in the works of D. R. Brillinger and T. Subba Rao.

Data under consideration

There are several fields of data where the linear model does not provide a satisfactory result. Our aim is to check the linearity of the data, and in case of nonlinearity, to show the higher order spectral properties and the use of bilinear fitting. Each of the data sets below are found on the Internet.

The S&P 500 Index

The primary objective of the Standard & Poor's 500 Composite Stock Price Index, known as the S&P 500, is to be the performance benchmark for the U.S. stock market. The Index is a market value-weighted index (shares outstanding times stock price) in which each company's influence in Index performance is directly proportional to its market value. The origins of the S&P 500 Index go back to 1923 when Standard & Poor introduced a series of indices that included 233 companies and covered 26 industries. The Index, as it is now known, was introduced in 1957. Today, the S&P 500 encompasses 500 companies, representing 90 specific industry groups. The Index is widely regarded as the standard for broad stock market performance The data of S&P 500 Index was found at the Web site of the Chadwick Investment Group by the address

http://chdwk.com/stock.html

among the Historical Stock Price Data. More information about the Index is listed at

http://www.cftech.com/BrainBank/FINANCE/SandPIndexCalc.html.

Recently, it has been shown [80] the non-Gaussianity of this index, and it was also pointed out that the probability density functions of GARCH(1,1) models are quite different from that observed data.

IBM stock prices

The data of the IBM stock prices come also from the Historical Stock Price Data library of the above named web site. It has been mentioned by the classic book of time series by Box and Jenkins [14] and also by Tong [140].

Geomagnetic indices

K indices isolate solar particle effects on the earth's magnetic field. Over a 3-hour period, they classify into disturbance levels the range of variation of the more unsettled horizontal field component. Each activity level relates almost logarithmically to its corresponding disturbance amplitude. Three-hour indices discriminate conservatively between true magnetic field perturbations and the quiet-day variations produced by ionospheric currents. The A-index ranges from 0 to 400 and represents a K-value converted to a linear scale in gammas (nanoteslas)—a scale that measures equivalent disturbance amplitude of a station at which K=9 has a lower limit of 400 gammas. The subscript p means planetary and designates a global magnetic activity index. The following 13 observatories, which lie between 46 and 63 degrees north and south geomagnetic latitude, now contribute to the planetary indices: Lerwick (UK), Eskdalemuir (UK), Hartland (UK), Ottawa (Canada), Fredericksburg, Virginia (USA), Meanook (Canada), Sitka, Alaska (USA), Eyrewell (New Zealand), Canberra (Australia), Lovo (Sweden), Brorfelde (Denmark), Wingst and Niemegk (Germany). For details see at National Geophysical Data Center

http://www.ngdc.noaa.gov/wdc/.

The aa-index is a simple global index of magnetic activity it is produced in France from the K indices of two nearly antipodal magnetic observatories in England and Australia. This index aa, is the 3-hourly equivalent amplitude antipodal index. Daily average aa may be derived similarly to ap. A historical advantage to using aa is that these indices have been extended back in time through scaling of magnetic activity from magnetograms of earlier observations. The aa indices are derived from 1868 to the present.

Magnetic field data

An example of bilinear systems comes from the nuclear magnetic resonance (NMR) spectroscopy studying the response of a changing magnetic field. The phenomenon is quantum mechanical and the concerning equation is bilinear it is called as Bloch equation. Brillinger, [22], [26], considered the analysis procedures of NMR spectroscopy when both the input and the output of system are observed. He estimated the unknown parameters of the bilinear equation. The data we consider here is a component of the multi-resolution magnetic field of the sun, measured by a spacecraft called Ulysses. The COHOWeb (http://nssdc.gsfc.nasa.gov/) provides access to hourly resolution magnetic field and plasma data from each of several heliospheric spacecraft. The hourly averages of parameters for the interplanetary magnetic field between October 25, 1990 and June 30, 1997 were chosen among the data available at the COHOWeb site. The principal investigator of magnetic field data was Dr. A. Balogh, Imperial College, London, UK. Three components of the magnetic field hour average are the following:

Magnetic field hour average of R component (nT).
Magnetic field hour average of T component (nT).
Magnetic field hour average of N component (nT).

The RTN system is fixed at a spacecraft (or the planet). The R axis is directed radially away from the sun; the T axis is the cross product of the solar rotation axis and the R axis, and the N axis is the cross product of R and T.

Data sets for the paper "A Single-Blind Controlled Competition Among Tests for Nonlinearity and Chaos" [9]

The data is simulated data, produced from five different generating models. One model, and hence two of the data sets, is purely deterministic (and chaotic). The other four models, and hence eight of the data sets, are stochastic processes, in which the randomness was produced by Monte Carlo methods. One of the stochastic processes was linear, while the other three were nonlinear, but not chaotic. The data were generated at Washington University in St. Louis, see http://wuecon.wustl.edu/~barnett/.

Acknowledgments

- The author wishes to thank David R. Brillinger and T. Subba Rao for several helpful discussions.
- Special thanks go to my colleagues Endre Iglói, Márton Ispány, and János Máth for joint work.
- The author highly appreciates the work of several research centers not only for measuring and analyzing the data but for providing them on-line for further investigations for all the Internet community, in particular the National Space Science Data Center NASA, USA; Chadwick Investment Group, USA; International Service of Geomagnetic Indices, CETP, France; University of Goettingen, Germany and professor William Barnett, Washington University in St. Louis, USA.
- This research is partially supported by the Hungarian National Science Foundation OTKA No. T 019501.
- The author wishes to thank the referees and Gy. Pap and L. Szeidl for their comments and suggestions.

Notations

The following notations are used

1. $\vec{1}$ denotes a row vector having all ones in its coordinates, i.e., $\vec{1} = (1, 1, ..., 1)$ with appropriate dimension.

2. Capitals X, Y, $Z\ldots$ stand for random variables.

3. Subscripting. Put $(1:n) = (1, 2, ..., n)$. The vector $(X_1, X_2, ..., X_n)$ will be denoted by $X_{(1:n)}$. In general if the subscript is a set (ordered) of natural numbers, say $a_{(1:n)} = (a_1, a_2, ..., a_n)$, then $X_{a_{(1:n)}}$ denotes a vector with components indexed by the elements of that set, i.e. $(X_{a_1}, X_{a_2}, ..., X_{a_n})$. In general if the subscript is a set (ordered) of natural numbers, say $K = \{a_1, a_2, ..., a_n\}$, then X_K denotes a vector with components indexed by the elements of that set, i.e. $(X_{a_1}, X_{a_2}, ..., X_{a_n})$. It sholuld cause no confusion to denote X_K also by $X_{a_{(1:n)}}$.

4. Let $\mathcal{K}$ denote some set of index sets, i.e. $\mathcal{K} = \{K, M, N\}$ where say $K = a_{(1:k_1)}$, $M = b_{(1:k_2)}$, $N = c_{(1:k_3)}$. The $X_{\mathcal{K}}$ denotes the vector of products with respect to the subsets of $\mathcal{K}$, e.g.,

$$X_{\mathcal{K}} = (\prod_{j=1}^{k_1} X_{a_j}, \prod_{j=1}^{k_2} X_{b_j}, \prod_{j=1}^{k_3} X_{c_j}).$$

5. The product and the sum associated with an index set have the following shorter notation

$$\Pi X_{a_{(1:n)}} = \prod_{j=1}^{n} X_{a_j}, \quad \Sigma X_{a_{(1:n)}} = \sum_{j=1}^{n} X_{a_j}.$$

6. $(a_{(1:n)}, x_{(1:n)})$ denotes the usual inner product of vectors, i.e.,

$$(a_{(1:n)}, x_{(1:n)}) = \sum_{k=1}^{n} a_k x_k.$$

7. Repetition. A vector having the same components, $(X_1, X_1, X_2, X_2, X_2) = X_{(1,1,2,2,2)}$, say, will be denoted by $X_{(1:2)[2,3]}$, as well. In general, the set with natural numbers in brackets [ ] denotes the number of the same components of the previous ordered set. In that sense

$$(X_1, X_1, X_2, X_2, X_2, X_3, X_3) = X_{\{1,1,2,2,2,3,3\}} = X_{(1:3)[2,3,2]}.$$

8. Exponent.

$$(X_1, X_2, ..., X_n)^{(t_1, t_2, ..., t_n)} = X_{(1:n)}^{t_{(1:n)}} = \prod_{k=1}^{n} X_k^{t_k},$$

i.e., the exponent of a vector by a vector with the same dimension is the product of the exponents, following this role

$$X_{(1:n)}^{r\vec{1}} = X_{(1:n)}^{(r,r,...,r)} = \prod\nolimits_{k=1}^{n} X_k^r.$$

9. Permutations. $\mathfrak{P}_n$ denotes the set of all permutations of the numbers $(1:n) = (1, 2, \ldots, n)$, if $\mathfrak{p} \in \mathfrak{P}_n$ then $\mathfrak{p}(1:n) = (\mathfrak{p}(1), \mathfrak{p}(2), \ldots, \mathfrak{p}(n))$.

10. Partitions. The set of all partitions of the numbers $(1:n) = (1, 2, \ldots, n)$, is denoted by $\mathcal{P}_{(1:n)}$, if $\mathcal{L} \in \mathcal{P}_{(1:n)}$ then $\mathcal{L} = \{K_1, K_2, \ldots, K_k\}$ such that $K_1, K_2, \ldots, K_k$ are disjoint and $\cup K_j = \{1, 2, \ldots, n\}$. In particular the set of all partitions into pairs of the set $(1, 2, ...n)$ is denoted by $\mathcal{P}_{(1:n)}^{\text{II}}$ and the set of all partitions having one or two elements is $\mathcal{P}_{(1:n)}^{\text{I,II}}$. $\mathcal{K}^{\text{II}}$ and $\mathcal{K}^{\text{I,II}}$ are the elements of $\mathcal{P}_{(1:n)}^{\text{II}}$ and $\mathcal{P}_{(1:n)}^{\text{I,II}}$, respectively.

11. Gaussian characteristic function is denoted by Ψ.

12. Gaussian distribution function is denoted by G.

13. Complex Gaussian stochastic spectral measure is denoted by W.

14. Spectral measure is Φ.

15. Cumulant spectral density is denoted by S.

16. $\overline{L_{\Phi}^{n}}$ and $\widetilde{L_{\Phi}^{n}}$ are Hilbert spaces.

17. Transfer functions are denoted by h, $g \ldots$.

18. $\mathcal{B}$ is the Borel σ-algebra with elements A, $B, \ldots$.

19. H_n is Hermite polynomial of order n.

20. $\mathcal{X}$ is Gaussian system if any subset of the elements of $\mathcal{X}$ contains jointly Gaussian random variables.

21. $\mathfrak{L}^2(\mathcal{X})$ is the Hilbert space of all random variables depending on the Gaussian system $\mathcal{X}$, see page 7.

22. $L_k^2(\mathcal{X})$ is the Hilbert space generated linearly by all possible Hermite polynomials with order k of the Gaussian system $\mathcal{X}$.

23. The time shift operator U_s for a stationary X_t, is defined by the equality $\mathsf{U}_s X_t = X_{t+s}$ for every s, t and is extended to a group of unitary transformations over $L_1^2(\mathcal{X})$.

24. The covariance and the second order cumulant of random variables Z_1, Z_2 with complex values are different

$$\begin{aligned}
\operatorname{Cov}(Z_1, Z_2) &= \operatorname{E}(Z_1 - \operatorname{E} Z_1)\overline{(Z_2 - \operatorname{E} Z_2)}, \\
\operatorname{Cum}(Z_1, Z_2) &= \operatorname{E}(Z_1 - \operatorname{E} Z_1)(Z_2 - \operatorname{E} Z_2).
\end{aligned}$$

25. $\bmod (1)$ denotes a relation on real numbers such that $x = y \bmod (1)$ if the fracional parts of x and y are equal.

26. $\delta_1(\cdot)$ is Dirac delta with periodic extension of $\bmod (1)$, called also Dirac comb.

27. $\delta_{\{\}}$ is Kronecker delta.

28. $\chi_A(\omega) = 1$ if $\omega \in A \bmod (1)$ and zero otherwise.

29. The Fock space is denoted by $\mathsf{Exp}\left(\widetilde{L_{\Phi}^{2}}\right)$.

30. $z_k = e^{i2\pi\omega_k}$, and by #8 above $z_{(1:n)}^{t\vec{1}} = e^{i2\pi t \sum_{k=1}^{n} \omega_k}$.

31. w_t, $t \in \mathbb{Z}$, is standard Gaussian white noise, e.g., w_t is an independent Gaussian series with mean 0 and variance 1.

32. $\mathcal{N}(\mu, \sigma^2)$ denotes the family of Gaussian random variables with mean μ and variance σ^2.

33. The semifactorial is $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$.

34. Symmetrized version of f_n is defined by

$$\begin{aligned}\operatorname{sym} f_n\left(\omega_{(1:n)}\right) &= f_n^{\mathfrak{P}}\left(\omega_1, \omega_2, \ldots, \omega_n\right) = \widetilde{f_n}\left(\omega_1, \omega_2, \ldots, \omega_n\right) \\ &= \frac{1}{n!} \sum_{\mathfrak{p} \in \mathfrak{P}_n} f_n\left(\omega_{\mathfrak{p}(1)}, \omega_{\mathfrak{p}(2)}, \ldots, \omega_{\mathfrak{p}(n)}\right).\end{aligned}$$

35. Symmetrized version by the variables $\omega_{(1:m)}$, where $n \leq m$, of a function f_n of variables $\omega_{(1:n)}$ is denoted by $\underset{\omega_{(1:m)}}{\operatorname{sym}} f_n$ and defined by

$$\begin{aligned}f_m^{\mathfrak{P}}\left(\omega_1, \omega_2, \ldots, \omega_n\right) &= \underset{\omega_{(1:m)}}{\operatorname{sym}} f_n\left(\omega_1, \omega_2, \ldots, \omega_n\right) \\ &= \frac{1}{m!} \sum_{\mathfrak{p} \in \mathfrak{P}_m} f_n\left(\omega_{\mathfrak{p}(1)}, \omega_{\mathfrak{p}(2)}, \ldots, \omega_{\mathfrak{p}(n)}\right).\end{aligned}$$

36. $\hat{f}(u_1, u_2)$ is the Fourier transform of $f(x_1, x_2)$, i.e.,

$$\hat{f}(u_1, u_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) exp\{-i(x_1 u_1 + x_2 u_2)\}\, dx_1\, dx_2.$$

37. Restricted Fourier transforms of cumulants of jointly stationary time series X_t, Y_t, and V_t

$$S_{X,Y}^{+}(z) = \sum_{s=1}^{\infty} \operatorname{Cum}\left(X_{t+s}, Y_t\right) z^{-s},$$

$$S_{X,Y,V}^{+}(z_1, z_2) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \operatorname{Cum}(X_{t+k+l}, Y_{t+l}, V_t) z_1^{-k-l} z_2^{-l}.$$

38. $S_{2,X}^{*n}$ denotes the n^{th} order convolution of $S_{2,X}$, i.e.,

$$\begin{aligned}S_{2,X}^{*n}(\omega) = \int_{[0,1]^{n-1}}^{2} & S_{2,X}\left(\omega - \omega_2\right) \prod_{k=2}^{n-1} S_{2,X}\left(\omega_k - \omega_{k+1}\right) \\ & \times S_{2,X}\left(\omega_n\right) d\omega_{(2:n)}. \qquad (1)\end{aligned}$$

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