NONLINEA
DIFFERENTIAL
EQUATIONS

RAIMOND A. STRUBLE

Professor of Applied Mathematics
North Carolina State College

RAIMOND A. STRUBLE

PROFESSOR OF APPLIED MATHEMATICS
NORTH CAROLINA STATE COLLEGE

NONLINEAR DIFFERENTIAL EQUATIONS

1962

McGRAW-HILL BOOK COMPANY, INC.

NEW YORK TORONTO LONDON

NONLINEAR DIFFERENTIAL EQUATIONS

 Library of Congress Catalog Card Number 61-14358

62246

PREFACE

This book has been written, first of all, to serve as a text for a one-semester advanced undergraduate or beginning graduate course in nonlinear differential equations. However, since it was prepared with the needs of the applied mathematician, engineer, and physicist specifically in mind, it should also prove useful as a reference text for scientists and engineers working in applied fields. This is not to suggest that the book is anything but a mathematics text; rather, it is to suggest that we must recognize and then compensate for the limited mathematical experience of many who today encounter and ponder nonlinear problems. There is nothing new in this approach, but it is seldom seen in mathematical works written for those with the high level of knowledge and achievement expected here. For example, it may appear inconsistent to place side by side elementary discussions of standard mathematical notation and advanced analytical techniques, but the alternatives are either to accept a mathematically unsatisfactory and incomplete job or to continue to deny a vast audience the genuine fruits of this vital and dynamic subject.

It has been the intention of the author to provide for rapid (though modest) contact with a majority of the mathematically significant concepts of nonlinear differential equations theory without overburdening the reader with a lot of loose ends. A concerted attempt has been made for brevity of treatment (consistent with mathematically sound principles) and simplification of concepts. Thus, it has been the further intention of the author to err (if he must) by recording all too little, rather than too much, and by oversimplifying, rather than overgeneralizing. The resultant shortcomings of this approach may, perhaps, be compensated by the multitude of exercises which form an integral part of the text. These contribute limited amounts of auxiliary, though sometimes essential, material, prepare the reader for subsequent work, and provide a running criticism of the text material itself. For the student, the last function of the exercises is by far the most important, and it is questionable if one can appreciate the real flavor of the work without careful note of this fact. Examples appear throughout the text and in the lists of exercises. These also contribute additional material, but more often than not, serve to illustrate the theorems, important concepts, or merely the notation and, in doing so, provide a link with more practical aspects of the theory. Chapter 8, which consists entirely of examples, stands in marked contrast to the theoretical pattern established in the earlier chapters. The asymptotic method illustrated therein is eminently practical and should dispel the notion that a variety of specialized techniques is required for treating traditional problems relating to linear and nonlinear oscillations.

The chapter and section titles are a sufficient indication of the total content. Though the chapter material represents a connected account of many areas of interest, the chapters themselves are not significantly interdependent and represent more or less distinct blocks of the total structure. A well-informed teacher may readily expand the content of any one chapter, drop or replace a chapter which, for example, might represent old material for a select audience, or insert a particular text chapter into another course or seminar. Only a few of the more pertinent references are given throughout the text. The list of general references includes excellent bibliographical sources, as well as other related information.

The starting point of this book was a set of lecture notes prepared for (and during) an internal seminar held at The Martin

Company, Denver Division. I wish to record here my appreciation to The Martin Company for providing me with the opportunity to participate in this seminar and with excellent secretarial help during the preparation of the lecture notes. I should like to thank John E. Fletcher and Steve M. Yionoulis for valuable assistance in the preparation of the manuscript and Mary Sue Davis for a superb job in typing the manuscript. As a student, I was fortunate to inherit from my teachers, especially Ky Fan, Joseph P. LaSalle, Karl Menger, and Arnold E. Ross, some of the rich traditions and finer things in mathematics. I sincerely hope that through this book I will share with future students at least a small part of this inheritance.

Raimond A. Struble

CONTENTS

Chapter 1

PRELIMINARY CONSIDERATIONS

1. *Linear Second-order Equations*

Consider the differential equation

$$\frac{d^2x}{dt^2} + x = 0 \tag{1}$$

which leads to the simple harmonic motion

$$x = A \sin (t + \Phi) \tag{2}$$

for arbitrary (constant) A and Φ. Let us introduce a second variable

$$y = \frac{dx}{dt} = A \cos (t + \Phi) \tag{3}$$

so that (2) and (3) together define the circle $x^2 + y^2 = A^2$ in parametric form with t as parameter. The solution in the xy plane is viewed, therefore, as a circle of radius $|A|$ centered at the origin.

A solution curve, viewed in the xy plane, is called a *trajectory*, and the xy plane itself is called the *phase plane*. A trajectory is oriented by the parameter t, and the direction of increasing t is indicated as in Fig. 1 by arrowheads. Note that from the definition of y, the arrowheads necessarily point toward positive x above the x axis and toward negative x below the x axis. A clockwise motion is thus indicated. The trivial solution of (1) corresponds to the origin $x = 0$, $y = 0$ and is called a *singular solution* or *point solution*. It represents a position of equilibrium. In this case there is but one position of equilibrium, and it is called a *center*, since all near trajectories are closed paths. Closed paths generally (but not always) correspond to periodic solutions, while periodic solutions always lead to trajectories which are closed paths.

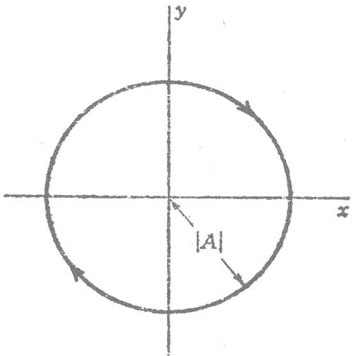

Figure 1

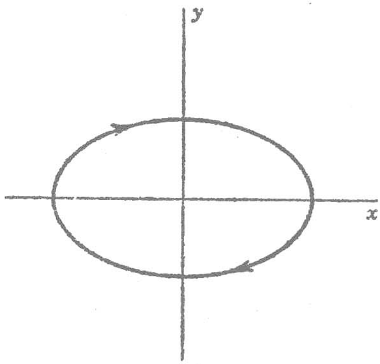

Figure 2

Let us now consider the trajectories defined by the equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0 \tag{4}$$

where each of k and ω is a constant. Without damping, i.e., $k = 0$, each trajectory of (4) is an ellipse (see Fig. 2). However, with damping, the trajectories are modified considerably. The nature of a solution depends upon the characteristic roots,

$$\lambda_1 = -\frac{k}{2} - \sqrt{\left(\frac{k}{2}\right)^2 - \omega^2} \qquad \lambda_2 = -\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 - \omega^2}$$

We shall examine the various cases in turn.

CASE 1: $\omega^2 > (k/2)^2$

Let $\omega_1 = \sqrt{\omega^2 - (k/2)^2}$ so that $\lambda_{1,2} = -k/2 \pm i\omega_1$. The general solution is well known, namely, $x = Ae^{-kt/2} \sin(\omega_1 t + \Phi)$ for arbitrary A and Φ. In this case

$$y = -\frac{k}{2} Ae^{-kt/2} \sin(\omega_1 t + \Phi) + \omega_1 Ae^{-kt/2} \cos(\omega_1 t + \Phi)$$

Let us introduce new dependent variables

$$\begin{aligned} u &= \omega_1 x = \omega_1 Ae^{-kt/2} \sin(\omega_1 t + \Phi) \\ v &= y + \frac{k}{2} x = \omega_1 Ae^{-kt/2} \cos(\omega_1 t + \Phi) \end{aligned} \tag{5}$$

If we interpret the solution as a trajectory in the uv plane, we obtain a spiral. Indeed from (5) we have

$$\rho^2 = u^2 + v^2 = \omega_1^2 A^2 e^{-kt}$$

and

$$\frac{u}{v} = \tan(\omega_1 t + \Phi)$$

Thus, for example, if $k > 0$, ρ^2 decreases monotonically as t increases, while the ratio u/v varies periodically with t (see Fig. 3). Again the motion is clockwise, although in this case v

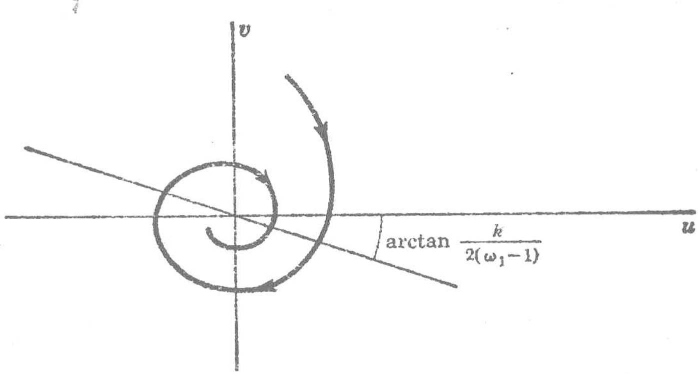

Figure 3

is not du/dt. In fact, u and v satisfy the equations

$$\begin{aligned}\frac{du}{dt} &= -\frac{k}{2}u + \omega_1 v \\ \frac{dv}{dt} &= -\omega_1 u - \frac{k}{2}v\end{aligned} \tag{6}$$

For $k < 0$, the trajectories spiral clockwise away from the origin. We note that the uv origin corresponds to the xy origin, i.e., the position of equilibrium. It is called a *focus* since near trajectories spiral either to or away from it.

This rather simple picture of the trajectories has been obtained through the use of the transformation (5). The latter is a *linear transformation* of the form

$$\begin{aligned}u &= b_{11}x + b_{12}y \\ v &= b_{21}x + b_{22}y\end{aligned} \tag{7}$$

If the determinant

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

is different from zero, then the mapping (7) is a one-to-one mapping of the xy plane onto the uv plane. Such transformations have the following important properties:

a. The origin maps to the origin.
b. Straight lines map to straight lines.
c. Parallel lines map to parallel lines.
d. The spacings of parallel lines remain in proportion.

These hold either for the mapping from the xy plane to the uv plane or for the inverse mapping from the uv plane to the xy plane. Thus an equilateral rectangular grid work will, in general, map onto a skewed grid work with different but uniform spacing in each of the two skewed directions. Many qualitative features of the trajectories are invariant under such transformations. For example, the logarithmic spiral in Fig. 3 is the image, under the linear transformation (5), of the distorted spiral in Fig. 4.

Some quantitative information may be obtained as follows. Let us consider the linear transformation

$$\begin{aligned} u &= \omega_1 x \\ v &= \frac{k}{2}x + y \end{aligned} \tag{8}$$

as a mapping from the two-dimensional *xy vector space* onto the two-dimensional *uv vector space*. Using rectangular cartesian

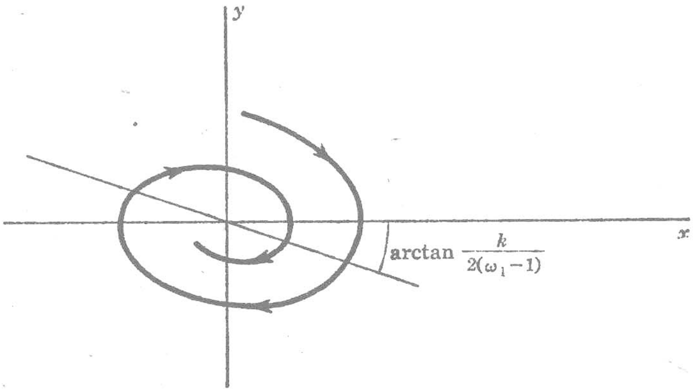

Figure 4

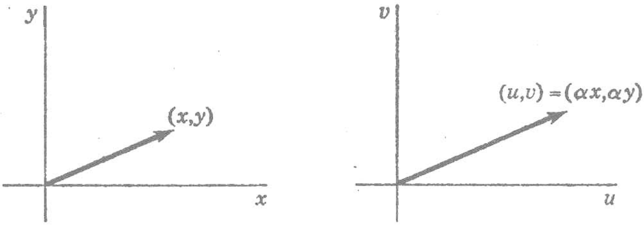

Figure 5

representation, as in Fig. 5, each *xy* vector may be identified with its end point (x,y) and its image vector under the transformation (8) by the end point (u,v) in the *uv* plane. We ask the following question: Are there any vectors in the *xy* plane which do *not* rotate under the transformation (8)? Such vectors are called *eigenvectors of the linear transformation.* If (u,v) is parallel to (x,y), the ratios v/u and y/x are equal, or what is the same, there exists a number α, called an *eigenvalue* of (8), for which

$$\begin{aligned} u &= \alpha x \\ v &= \alpha y \end{aligned} \tag{9}$$

The eigenvalue itself is a "stretching" factor, since the length of the uv vector is $|\alpha|$ times the length of the corresponding xy vector. But from (8) and (9) we conclude that necessarily

$$\begin{aligned} \omega_1 x &= \alpha x \\ \frac{k}{2} x + y &= \alpha y \end{aligned}$$

or, what is the same,

$$\begin{aligned} (\omega_1 - \alpha)x &= 0 \\ \frac{k}{2} x + (1 - \alpha)y &= 0 \end{aligned} \tag{10}$$

In general, there are two nontrivial solutions of (10):

$$\alpha = 1 \qquad x = 0 \qquad (y, \text{ arbitrary}) \tag{11}$$

$$\alpha = \omega_1 \qquad \frac{k}{2} x + (1 - \omega_1)y = 0 \tag{12}$$

corresponding to the two eigenvalues $\alpha = 1$, $\alpha = \omega_1$. If $\omega_1 = 1$, (11) and (12) are one and the same. More generally, the first asserts that vectors parallel to the y axis are not rotated, while the second asserts that vectors with slope equal to $k/2(\omega_1 - 1)$ are not rotated. Further, since $\alpha = 1$ in (11) and $\alpha = \omega_1$ in (12), we conclude that the lengths of the vectors parallel to the y axis remain unchanged while the lengths of the vectors with slope equal to $k/2(\omega_1 - 1)$ are stretched by the factor ω_1. Thus the distorted spiral in Fig. 4 is obtained from the logarithmic spiral in Fig. 3 by moving the intercepts with the line $u = [k/2(\omega_1 - 1)]v$ (shown for $\omega_1 < 1$) outward in proportion to $1/\omega_1$, while leaving the intercepts on the v axis as they are.

CASE 2: $(k/2)^2 > \omega^2$

Let us write the single equation (4) as the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\omega^2 x - ky \end{aligned} \tag{13}$$

In this case, we seek a linear transformation (7) such that the system (13) becomes

$$\begin{aligned} \frac{du}{dt} &= \alpha_1 u \\ \frac{dv}{dt} &= \alpha_2 v \end{aligned} \tag{14}$$

for suitable constants α_1, α_2. The system (14) is "uncoupled" and the solutions may be obtained immediately.

Applying the transformation (7) to (14) and using (13), we obtain

$$\begin{aligned} b_{11}y + b_{12}(-\omega^2 x - ky) &= \alpha_1(b_{11}x + b_{12}y) \\ b_{21}y + b_{22}(-\omega^2 x - ky) &= \alpha_2(b_{21}x + b_{22}y) \end{aligned} \tag{15}$$

If these equations are to hold identically in x and y, then the total coefficient of each of x and y must vanish. We consider, therefore, the two sets of equations

$$\begin{aligned} -\omega^2 b_{12} &= \alpha_1 b_{11} \\ b_{11} - kb_{12} &= \alpha_1 b_{12} \end{aligned} \tag{16}$$

and

$$\begin{aligned} -\omega^2 b_{22} &= \alpha_2 b_{21} \\ b_{21} - kb_{22} &= \alpha_2 b_{22} \end{aligned} \tag{17}$$

The first of (16) may be written

$$\frac{b_{12}}{b_{11}} = -\frac{\alpha_1}{\omega^2} \tag{18}$$

and the second then becomes

$$1 + k\frac{\alpha_1}{\omega^2} = -\frac{\alpha_1^2}{\omega^2} \tag{19}$$

Similarly, the two equations of (17) yield

$$\frac{b_{22}}{b_{21}} = -\frac{\alpha_2}{\omega^2} \tag{20}$$

and

$$1 + k\frac{\alpha_2}{\omega^2} = -\frac{\alpha_2^2}{\omega^2} \tag{21}$$

Equations (19) and (21) are merely versions of the characteristic equation $\lambda^2 + k\lambda + \omega^2 = 0$. Thus each of α_1 and α_2 must be a characteristic root.

With α_1 and α_2 determined, Eqs. (18) and (20) determine the ratios b_{12}/b_{11} and b_{22}/b_{21}. For convenience, we may choose $b_{11} = b_{21} = \omega^2$, and the desired linear transformation may be expressed

$$\begin{aligned} u &= \omega^2 x - \lambda_1 y \\ v &= \omega^2 x - \lambda_2 y \end{aligned} \tag{22}$$

We note that the determinant

$$\begin{vmatrix} \omega^2 & -\lambda_1 \\ \omega^2 & -\lambda_2 \end{vmatrix} = \omega^2(\lambda_1 - \lambda_2)$$

is different from zero, since $\lambda_1 \neq \lambda_2$. Thus (22) is nonsingular.

In the new variables, the solutions are given by (14) with $\alpha_1 = \lambda_1$ and $\alpha_2 = \lambda_2$. We have

$$\begin{aligned} u &= u_0 e^{\lambda_1 t} \\ v &= v_0 e^{\lambda_2 t} \end{aligned} \tag{23}$$

for arbitrary u_0 and v_0. For $k > 0$, both characteristic roots are negative and so each trajectory in the uv plane approaches the origin as $t \to \infty$. Further, the ratio

$$\frac{u}{v} = \frac{u_0}{v_0} e^{(\lambda_1 - \lambda_2)t}$$

approaches zero as $t \to \infty$, since $\lambda_1 - \lambda_2 = -2\sqrt{(k/2)^2 - \omega^2}$. Thus the trajectories are asymptotic to the v axis. From (23) we have

$$\frac{u^{\lambda_2}}{v^{\lambda_1}} = \frac{(u_0 e^{\lambda_1 t})^{\lambda_2}}{(v_0 e^{\lambda_2 t})^{\lambda_1}} = \frac{u_0^{\lambda_2}}{v_0^{\lambda_1}} = \text{const}$$

so that the uv trajectories lie along the curves $u = (\text{const})v^{\lambda_1/\lambda_2}$. The singular solution in this case is called a *node*. (All near trajectories tend to or away from a node without spiraling.) For $k < 0$, the solution curves are somewhat similar to those illustrated in Fig. 6. However, since both characteristic roots are

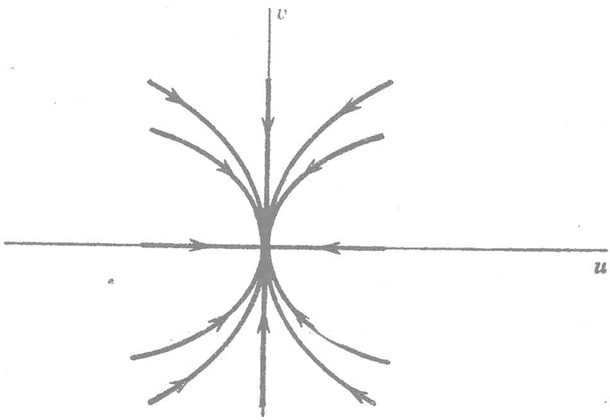

Figure 6

then positive, the arrows must be reversed and the labels on the two axes must be interchanged.

The trajectories in the phase plane are also qualitatively similar to those shown in Fig. 6 but will appear to be rotated and stretched. The eigenvalues of the transformation (22) are roots of the equation

$$\begin{vmatrix} \omega^2 - \lambda & -\lambda_1 \\ \omega^2 & -\lambda_2 - \lambda \end{vmatrix} = \lambda^2 + (\lambda_2 - \omega^2)\lambda + (\lambda_1 - \lambda_2)\omega^2 = 0$$

and the eigenvectors (i.e., invariant directions) could be obtained as before. However, in this case it is probably more important to know what happens to the u axis and v axis under (22). The