Advances in
Discrete Mathematics and Applications
Volume 4

Periodicities in Nonlinear Difference Equations

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G. Ladas

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Periodicities in Nonlinear Difference Equations

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To Lina and Maureen

Preface

"**Sharkovsky's Theorem**," the "**Period 3 implies chaos**" result of Li and Yorke, and the "**(3x+1)-Conjecture**" are beautiful and deep results showing the rich periodic character of first-order, non-linear difference equations. During the last ten years, we have been fascinated discovering non-linear difference equations of order greater than one which for certain values of their parameters have one of the following characteristics:

1. **Every solution of the equation is *periodic* with the same period.**
2. **Every solution of the equation is *eventually periodic* with a prescribed period.**
3. **Every solution of the equation *converges* to a periodic solution with the same period.**

Our goal in this monograph is to bring to the attention of the mathematical community these equations, together with **some thought-provoking questions and a great number of open problems and conjectures** which we strongly believe are worthy of investigation.

We would also like to begin the investigation of the global character of solutions of these equations for other values of their parameters and to attempt to see a more complete picture of the global behavior of their solutions.

We believe that the results in this monograph place a few more stones in the foundation of the "Basic Theory of Nonlinear Difference Equations of Order Greater Than One," where at the beginning of the third millennium, we surprisingly know so little.

Acknowledgments

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Contents

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Chapter 1

PRELIMINARIES

1.1 Introduction

In this chapter we present some definitions and state some known results which will be useful in the subsequent chapters. For further details and additional references, see [5], [6], [33], [36], [37], [61], [62], [72], [101], [112], and [115].

The reader may simply glance at the results in this chapter and return for the details when they are needed in the sequel. In this way, this is a self-contained monograph, and the main prerequisite that the reader needs to understand the material in this book and to be able to attack the open problems and conjectures is a solid foundation in analysis.

1.2 Definitions of Stability and Linearized Stability

A *difference equation of order* $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) \quad , \qquad n = 0, 1, \ldots \tag{1.1}$$

where f is a continuous function which maps some set J^{k+1} into J. The set J is usually an interval of real numbers, or a union of intervals, but it may even be a discrete set such as the set of *integers* $\mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$.

A *solution* of Eq.(1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies Eq.(1.1) for all $n \geq 0$. If we prescribe a set of $(k+1)$ *initial conditions*

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in J$$

then

$$\begin{aligned} x_1 &= f(x_0, x_{-1}, \ldots, x_{-k}) \\ x_2 &= f(x_1, x_0, \ldots, x_{-k+1}) \\ &\vdots \end{aligned}$$

and so the solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) exists for all $n \geq -k$ and is uniquely determined by the initial conditions.

A solution of Eq.(1.1) which is constant for all $n \geq -k$ is called an *equilibrium solution* of Eq.(1.1). If

$$x_n = \bar{x} \qquad \text{for all} \qquad n \geq -k$$

is an equilibrium solution of Eq.(1.1), then $\bar{x}$ is called an *equilibrium point*, or simply an *equilibrium*, of Eq.(1.1).

DEFINITION 1.1 **(Stability)**

(i) We say that an equilibrium point $\bar{x}$ of Eq.(1.1) is locally stable *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1) with*

$$|\, x_{-k} - \bar{x} \,| + |\, x_{1-k} - \bar{x} \,| + \cdots + |\, x_0 - \bar{x} \,| < \delta,$$

then

$$|\, x_n - \bar{x} \,| < \varepsilon \qquad \text{for all} \qquad n \geq -k.$$

(ii) We say that an equilibrium point $\bar{x}$ of Eq.(1.1) is locally asymptotically stable *if $\bar{x}$ is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1) with*

$$|\, x_{-k} - \bar{x} \,| + |\, x_{-k+1} - \bar{x} \,| + \cdots + |\, x_0 - \bar{x} \,| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii) We say that the the equilibrium point $\bar{x}$ of Eq.(1.1) is a global attractor *if for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1), we have*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) We say that the the equilibrium point $\bar{x}$ of Eq.(1.1) is globally asymptotically stable *if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq.(1.1).*

(v) We say that the the equilibrium point $\bar{x}$ of Eq.(1.1) is unstable *if $\bar{x}$ is not locally stable.*

(vi) We say that the the equilibrium point $\bar{x}$ of Eq.(1.1) is a source *if there exists $r > 0$ such that for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) with*

$$0 < |\, x_{-k} - \bar{x} \,| + |\, x_{-k+1} - \bar{x} \,| + \cdots + |\, x_0 - \bar{x} \,| < r,$$

there exists $N \geq 1$ such that

$$|\, x_N - \bar{x} \,| > r.$$

Clearly a source is an unstable equilibrium point of Eq.(1.1).

Suppose f is continuously differentiable in some open neighborhood of $\bar{x}$. Let

$$p_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \ldots, \bar{x}) \qquad \text{for} \qquad i = 0, 1, \ldots, k$$

denote the partial derivative of $f(u_0, u_1, \ldots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(1.1). Then the equation

$$y_{n+1} = p_0 z_n + p_1 z_{n-1} + \cdots + p_k z_{n-k} \quad , \qquad n = 0, 1, \ldots \tag{1.2}$$

is called the *linearized equation of Eq.(1.1) about the equilibrium point* $\bar{x}$, and the equation

$$\lambda^{k+1} - p_0 \lambda^k - \cdots - p_{k-1}\lambda - p_k = 0 \tag{1.3}$$

is called the *characteristic equation of Eq.(1.2) about* $\bar{x}$.

The following well-known result, called the *Linearized Stability Theorem,* is very useful in determining the local stability character of the equilibrium point $\bar{x}$ of Eq.(1.1). See [6], [36], [62], and [72].

THEOREM 1.1

(The Linearized Stability Theorem)

Suppose f is a continuously differentiable function defined on some open neighborhood of $\bar{x}$. Then the following statements are true:

1. *If all the roots of Eq.(1.3) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq.(1.1) is locally asymptotically stable.*

2. *If at least one root of Eq.(1.3) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(1.1) is unstable.*

3. *If all the roots of Eq.(1.3) have absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(1.1) is a source.*

The equilibrium point $\bar{x}$ of Eq.(1.1) is called *hyperbolic* if no root of Eq.(1.3) has absolute value equal to one. If there exists a root of Eq.(1.3) with absolute value equal to one, then $\bar{x}$ is called *non-hyperbolic.*

The equilibrium point $\bar{x}$ of Eq.(1.1) is called a *sink* if every root of Eq.(1.3) has absolute value less than one. Thus a sink is locally asymptotically stable, but the converse need not be true.

The equilibrium point $\bar{x}$ of Eq.(1.1) is called a *saddle point equilibrium point* if it is hyperbolic, and if in addition, there exists a root of Eq.(1.3) with absolute value less than one and another root of Eq.(1.3) with absolute value greater than one. In particular, a saddle point equilibrium point is unstable.

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called *periodic with period* p (or a period-p solution) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \qquad \text{for all} \qquad n \geq -k. \tag{1.4}$$

We say that the solution is periodic with *prime period* p if p is the smallest positive integer for which Eq.(1.4) holds. In this case, a *p-tuple*

$$(x_{n+1},\ x_{n+2}, \ldots,\ x_{n+p})$$

of any p consecutive values of the solution is called a *p-cycle* of Eq.(1.1).

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called *eventually periodic with period p* if there exists an integer $N \geq -k$ such that $\{x_n\}_{n=N}^{\infty}$ is periodic with period p; that is,

$$x_{n+p} = x_n \qquad \text{for all} \qquad n \geq N.$$

The following lemma describes when a solution of Eq.(1.1) converges to a periodic solution of Eq.(1.1).

LEMMA 1.1

Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1), and let $p \geq 1$ be a positive integer. Suppose there exist real numbers $l_0, l_1, \ldots, l_{p-1} \in J$ such that

$$\lim_{n\to\infty} x_{pn+j} = l_j \qquad \textit{for all} \qquad j = 0, 1, \ldots, p-1.$$

Finally, let $\{y_n\}_{n=-k}^{\infty}$ be the period-p sequence of real numbers in J such that for every integer j with $0 \leq j \leq p-1$, we have

$$y_{pn+j} = l_j \qquad \textit{for all} \qquad n = 0, 1, \ldots .$$

Then the following statements are true:

1. *$\{y_n\}_{n=-k}^{\infty}$ is a period-p solution of Eq.(1.1).*
2. $\lim_{n\to\infty} x_{pn+j} = y_j \qquad$ *for every* $\qquad j \geq -k.$

PROOF It suffices to show that $\{y_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1). Note that for $j \geq 0$, we have

$$\begin{aligned} y_{j+1} &= \lim_{n\to\infty} x_{pn+j+1} = \lim_{n\to\infty} f(x_{pn+j}, x_{pn+j-1}, \ldots, x_{pn+j-k}) \\ &= f(y_j, y_{j-1}, \ldots, y_{j-k}). \end{aligned}$$

□

We now state Theorem 1.2 which explains the significance of Eq.(1.3) having root(s) with absolute value less than one and also root(s) with absolute value greater than one. We shall use Theorem 1.2 to show that when this occurs, even though the equilibrium $\bar{x}$ is unstable, there still exist non-trivial solutions which converge to it, and so in particular, there exist non-trivial, bounded solutions of Eq.(1.1).

Our presentation is extracted from the treatment of this topic found in [112].

We first need some notation.

Let V be a non-empty open subset of $\mathbf{R}^{k+1}$, and let $T : V \to R^{k+1}$ be a (not necessarily invertible) C^m map, where $1 \leq m \leq \infty$.

Let $\vec{p} \in V$ be an equilibrium point of T. Suppose that at least one eigenvalue of $J_T(\vec{p})$ has absolute value less than one and at least one eigenvalue of $J_T(\vec{p})$ has absolute value greater than one.

Given an open subset V_1 of V with $\vec{p} \in V_1$, the *local stable manifold* for $\vec{p}$ in the neighborhood V_1 is defined as follows:

$$\mathcal{S}(\vec{p}, V_1, T) = \{\vec{q} \in V_1 : T^n(\vec{q}) \in V_1 \text{ for all } n \geq 0 \text{ and}$$

$$\lim_{n\to\infty} \|T^n(\vec{q}) - \vec{p}\| = 0\}.$$

Define a *past history* of a point $\vec{q}$ to be a sequence of points $\{\vec{q}_{-n}\}_{n=0}^{\infty}$ such that $\vec{q}_0 = \vec{q}$ and $T(\vec{q}_{-n-1}) = \vec{q}_{-n}$ for all $n \geq 0$. The *local unstable manifold* for $\vec{p}$ in the neighborhood V_1 is defined as follows:

$$\mathcal{U}(\vec{p}, V_1, T) = \{\vec{q} \in V_1 : \text{there exists a past history } \{\vec{q}_{-n}\}_{n=0}^{\infty} \subset V_1 \text{ of } \vec{q} \text{ such that}$$

$$\lim_{n\to\infty} \|T^{-n}(\vec{q}) - \vec{p}\| = 0\}.$$

Let S be the eigen-space of $J_T(\vec{p})$ which corresponds to eigenvalues with absolute value less than one, and let U be the eigen-space of $J_T(\vec{p})$ which corresponds to eigenvalues with absolute value greater than one.

THEOREM 1.2

There exists an open subset V_1 of V with $\vec{p} \in V_1$ such that $\mathcal{S}(\vec{p}, V_1, T)$ and $\mathcal{U}(\vec{p}, V_1, T)$ are C^m manifolds. The tangent space of $\mathcal{S}(\vec{p}, V_1, T)$ at $\vec{p}$ is S, and the tangent space of $\mathcal{U}(\vec{p}, V_1, T)$ at $\vec{p}$ is U.

Theorem 1.2 can be extended in a straightforward fashion to the case where $\vec{p}$ is a periodic point of T.

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