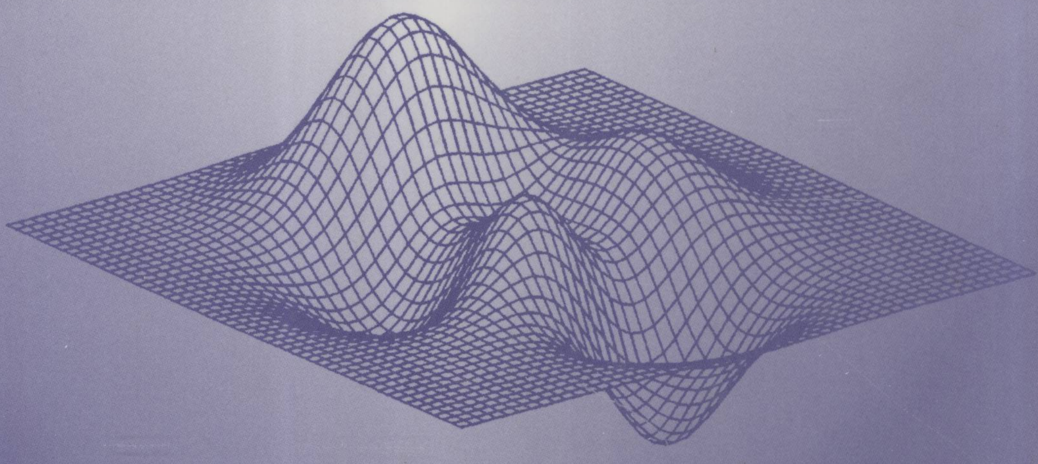
NONCONVEX OPTIMIZATION AND ITS APPLICATIONS

Generalized Convexity, Generalized Monotonicity and Applications

Edited by
Andrew Eberhard, Nicolas Hadjisavvas
and Dinh The Luc



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GENERALIZED CONVEXITY, GENERALIZED MONOTONICITY AND APPLICATIONS

Proceedings of the 7th International Symposium on Generalized Convexity and Generalized Monotonicity

Edited by

ANDREW EBERHARD
RMIT University, Australia

NICOLAS HADJISAVVAS
University of the Aegean, Greece

DINH THE LUC
University of Avignon, France

Springer

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GENERALIZED CONVEXITY, GENERALIZED MONOTONICITY AND APPLICATIONS

Nonconvex Optimization and Its Applications

Volume 77

Preface

In recent years there is a growing interest in generalized convex functions and generalized monotone mappings among the researchers of applied mathematics and other sciences. This is due to the fact that mathematical models with these functions are more suitable to describe problems of the real world than models using conventional convex and monotone functions. Generalized convexity and monotonicity are now considered as an independent branch of applied mathematics with a wide range of applications in mechanics, economics, engineering, finance and many others.

The present volume contains 20 full length papers which reflect current theoretical studies of generalized convexity and monotonicity, and numerous applications in optimization, variational inequalities, equilibrium problems etc. All these papers were refereed and carefully selected from invited talks and contributed talks that were presented at the 7th International Symposium on Generalized Convexity/Monotonicity held in Hanoi, Vietnam, August 27-31, 2002. This series of Symposia is organized by the Working Group on Generalized Convexity (WGGC) every 3 years and aims to promote and disseminate research on the field. The WGGC (http://www.genconv.org) consists of more than 300 researchers coming from 36 countries.

Taking this opportunity, we want to thank all speakers whose contributions make up this volume, all referees whose cooperation helped in ensuring the scientific quality of the papers, and all people from the Hanoi Institute of Mathematics whose assistance was indispensable in running the symposium. Our special thanks go to the Vietnam Academy of Sciences and Technology, the Vietnam National Basic Research Project "Selected problems of optimization and scientific computing" and the Abdus Salam International Center for Theoretical Physics at Trieste, Italy, for their generous support which made the meeting possible. Finally, we express our appreciation to Kluwer Academic Publishers for including this volume into their series. We hope that the volume will

be useful for students, researchers and those who are interested in this emerging field of applied mathematics.

ANDREW EBERHARD

NICOLAS HADJISAVVAS

DINH THE LUC

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I

INVITED PAPERS

Chapter 1

ALGEBRAIC DYNAMICS OF CERTAIN GAMMA FUNCTION VALUES

J.M. Borwein*
Research Chair, Computer Science Faculty,
Dalhousie University, Canada

K. Karamanos
Centre for Nonlinear Phenomena and Complex Systems,
Université Libre de Bruxelles, Belgium

Abstract We present significant numerical evidence, based on the entropy analysis by lumping of the binary expansion of certain values of the Gamma function, that some of these values correspond to incompressible algorithmic information. In particular, the value $\Gamma(1/5)$ corresponds to a peak of non-compressibility as anticipated on a priori grounds from number-theoretic considerations. Other fundamental constants are similarly considered.

This work may be viewed as an invitation for other researchers to apply information theoretic and decision theory techniques in number theory and analysis.

Keywords: Algebraic dynamics, symbolic dynamics.

MSC2000: 94A15, 94A17, 37Bxx, 11Yxx, 11Kxx

1. Introduction

Nature provides us with a wide variety of symbolic strings ranging from the sequences generated by the symbolic dynamics of nonlinear systems to RNA and DNA sequences or DLA patterns (*diffusion limited*

*email:jborwein@cs.dal.ca

aggregation patterns are a classical subject in Nonlinear Chemistry); see Hao (1994); Nicolis et al (1994); Schröder (1991).

Entropy-like quantities are a very useful tool for the analysis of such sequences. Of special interest are the *block entropies*, extending Shannon's classical definition of the entropy of a single state to the entropy of a succession of states (Nicolis et al (1994)). In particular, it has been shown in the literature that *scaling* the block entropies by length sometimes yields interesting information on the structure of the sequence (Ebeling et al (1991); Ebeling et al (1992)).

In particular, one of the present authors has derived an entropy criterion for the specialized, yet important algorithmic property of *automaticity* of a sequence. We recall that, a sequence is called *automatic* if it is generated by a finite automaton (the lowest level Turing machine). For more details about automatic sequences the reader is referred to Cobham (1972), and for their role in Physics to Allouche (2000).

This criterion is based on entropy analysis by *lumping. Lumping* is the reading of the symbolic sequence by 'taking portions' (see expression (1)), as opposed to *gliding* where one has essentially a 'moving frame'. Notice that gliding is the standard approach in the literature. Reading a symbolic sequence in a specific way is also called *decimation* of the sequence.

The paper is articulated as follows. In Section two we recall some useful facts. In Section three we present the mathematical formulation of the entropy analysis by lumping. In Section four we present our intuitive motivation based on algorithmic arguments while in Section five we present a central example of an automatic sequence, taken from the world of nonlinear Science, namely the Feigenbaum sequence. In Section six we present our main results. In Section seven we speak about automaticity and algorithmic compressibility measures. In section eight we analyse $\exp(\pi/\sqrt(2))$. Finally, in Section nine we draw our main conclusions and discuss future work.

2. Some definitions

We first recall some useful facts from elementary number theory. As is well known, *rational* numbers can be written in the form of a fraction p/q, where p and q are integers and *irrational* ones cannot take this form. The k-ary expansion of a rational number (for instance the decimal or binary expansion) is periodic or eventually periodic and conversely. Irrational numbers form two categories: *algebraic irrational* and *transcendental*, according to whether they can be obtained as roots of a polynomial with rational coefficients or not. The k-ary expansion of an irrational number is necessarily *aperiodic*. Note that transcendental numbers are *well approximated* by fractions. In 1874 G. Cantor showed that 'almost all' real numbers are transcendental.

A *normal* number in base $k \geq 2$ is a real number x such that, for each integer $d \geq 1$, each block of length d occurs in the k-ary expansion of x with (equal) asymptotic frequency $1/k^d$. A rational number is never normal, while there exist numbers which are normal and transcendental, like Champernowne's number. This number is obtained by concatenating the decimal expansions of consecutive integers (Champernowne (1933))

$$0.1234567891011121314...$$

and it is simultaneously transcendental and normal in base 10.

There is an important and widely believed conjecture, according to which all algebraic irrational numbers are believed to be normal. But present techniques fall woefully short on this matter, see Bailey et al (2004). It seems that E. Borel was the first who explicitly formulated such a conjecture in the early fifties (Borel (1950)). Actually, normality is not the best criterion to distinguish between algebraic irrational and transcendental numbers. In fact, there exist transcendental numbers which are normal, like Champernowne's number (Champernowne (1933), Chaitin (1994), Allouche (2000)) and probably π (Schröder (1991), Wagon (1985) Allouche (2000)). One of the first systematic studies towards this direction dates back to ENIAC also some fifty years ago (Metropolis et al (1950); Borwein (2003)). No truly 'natural' transcendental number has been shown to be normal in any base, hence the interest in computation.

3. Entropy analysis by lumping

For reasons both of completeness and for later use, we compile here the basic ideas of the method of entropy analysis by lumping. We consider a subsequence of length N selected out of a very long (theoretically infinite) symbolic sequence. We stipulate that this subsequence is to be read in terms of distinct 'blocks' of length n,

$$\cdots \underbrace{A_1 \ldots A_n}_{B_1} \underbrace{A_{n+1} \cdots A_{2n}}_{B_2} \cdots \underbrace{A_{jn+1} \cdots A_{(j+1)n}}_{B_{j+1}} \cdots$$

We call this reading procedure *lumping*. We shall employ lumping throughout the sequel. The following quantities characterize the information content of the sequence (Khinchin (1957); Ebeling et al (1991)).

i) The *dynamical (Shannon-like) block-entropy* for blocks of length n is given by

$$H(n) := - \sum_{(A_1,...A_n)} p^{(n)}(A_1, \ldots, A_n) \cdot \ln p^{(n)}(A_1, \ldots, A_n) \quad (1.1)$$

where the probability of occurrence of a block $A_1 \ldots A_n$, denoted $p^{(n)}(A_1, ..., A_n)$, is defined (when it exists) in the statistical limit as

$$p^{(n)}(A_1, \ldots, A_n) = \frac{\#\text{of blocks } A_1 \ldots A_n \text{ found when lumping}}{\text{Total } \# \text{ of blocks found}} \quad (1.2)$$

starting from the beginning of the sequence, and the associate entropy per letter

$$h^{(n)} = \frac{H(n)}{n}. \quad (1.3)$$

ii) The *conditional entropy* or entropy excess associated with the addition of a symbol to the right of an n-block

$$h_{(n)} = H(n+1) - H(n). \quad (1.4)$$

iii) The *entropy of the source* (a topological invariant), defined as the limit (if it exists)

$$h = \lim_{n \to \infty} h_{(n)} = \lim_{n \to \infty} h^{(n)} \quad (1.5)$$

which is the discrete analogue of metric or Kolmogorov entropy.

We now turn to the selection problem, that is to the possibility of emergence of some preferred configurations (blocks) out of the complete set of different possibilities. The number of all possible symbolic sequences of length n (*complexions* in the sense of Boltzmann) in a K-letter alphabet is

$$N_K = K^n. \quad (1.6)$$

Yet not all of these configurations are necessarily realized by the dynamics, nor are they equiprobable. A remarkable theorem due to McMillan (see Khinchin (1957)), gives a partial answer to the selection problem asserting that for stationary and ergodic sources the probability of occurrence of a block $(A_1, \ldots, A_n)$ is

$$p^{(n)}(A_1, \ldots, A_n) \sim e^{-H(n)} \quad (1.7)$$

for almost all blocks $(A_1, \ldots, A_n)$. In order to determine the abundance of long blocks one is thus led to examine the scaling properties of $H(n)$ as a function of n.

It is well known that numerically, block entropy is underestimated. This underestimation of $H(n)$ for large values of n is due to the simple fact that not all words will be represented adequately if one looks at long enough samples. The situation becomes more and more prominent for calculating $H(n)$ by 'lumping' instead of 'gliding'. Indeed in the case of 'lumping' an exponentially fast decaying tail towards value zero follows after an initial plateau.

Since the probabilities of the words of length m are calculated by their frequencies, i.e. $p_n = N_1/N_{[sample]}$ where $N_{[sample]}$ is the size of the available data-sample i.e. the length of the 'text' under consideration, then as $N_1 \to 0$ for long words, the block entropy calculated will reach a maximum value, its *plateau*, at

$$H_{MAX} = \log_{[K]}(N_{[sample]})$$

where K the length of the alphabet. Indeed, this corresponds to the maximum value of the entropy for this sample, given when

$$p^{(n)} = 1/N_{[sample]}.$$

This value corresponds also to an effective maximum word length

$$n_{max} = \ln N_{[sample]}$$

in view of eqs. (1), (6) and (7).

For instance, if we have a binary sequence with 10,000 terms, of course $b = 2$ and $N_{[sample]} = 10^4$. This way, the value of H_{MAX} can determine a safe border for finite size effects. In our case

$$H_{MAX} = n_{max} = \ln\left(10^4\right) = 9.2\ldots, \tag{1.8}$$

so that $n_{max} = 9$ and we can safely consider the entropies until $n = 8$.

After this small digression, we recall here the main result of the entropy analysis by lumping, see also Karamanos (2001b); Karamanos (2001c). Let m^k be the length of a block encountered when lumping, $H(m^k)$ the associated block entropy. We recall that, in view of a result by Cobham (Theorem 3 of Cobham (1972)), a sequence is called *m-automatic* if it is the image by a letter to letter projection of the fixed point of a set of substitutions of constant length m. A substitution is called *uniform* or *of constant length* if all the images of the letters have the same length. For instance, the Feigenbaum symbolic