CONTEMPORARY MATHEMATICS

397

# The Geometry of Riemann Surfaces and Abelian Varieties

III Iberoamerican Congress on Geometry
in Honor of Professor Sevín Recillas-Pishmish's 60th Birthday
June 8–12, 2004
Salamanca, Spain

José M. Muñoz Porras
Sorin Popescu
Rubí E. Rodríguez
Editors



American Mathematical Society

# CONTEMPORARY MATHEMATICS

397

# The Geometry of Riemann Surfaces and Abelian Varieties

III Iberoamerican Congress on Geometry
in Honor of Professor Sevín Recillas-Pishmish's 60th Birthday
June 8–12, 2004
Salamanca, Spain

José M. Muñoz Porras
Sorin Popescu
Rubí E. Rodríguez
Editors



**American Mathematical Society**
Providence, Rhode Island

This volume contains the proceedings of the III Iberoamerican Congress on Geometry in honor of Professor Sevín Recillas-Pishmish's 60th birthday, held at the University of Salamanca, Spain, from June 8–12, 2004, with support from Spain (University of Salamanca, DGI grant BFM2002-12288-E, Junta de Castilla y León and Caja Duero), the United States of America (NSF grant DMS-0342699) and Chile (Fondecyt grant 1030595).

2000 *Mathematics Subject Classification.* Primary 30F10, 14H30, 14H40; Secondary 14H10, 14H15, 14H37, 14K25, 14K05, 30F40.

---

**Library of Congress Cataloging-in-Publication Data**

Iberoamerican Congress on Geometry (3rd: 2004: Salamanca, Spain)
The geometry of Riemann surfaces and Abelian varieties: III Iberoamerican Congress on Geometry in honor of Professor Sevin Recillas-Pishmish's 60th birthday, June 8–12, 2004, Salamanca, Spain/ Jose M. Munoz Porras, Sorin Popescu, Rubi E. Rodriguez, editors.
p. cm. – (Contemporary mathematics, ISSN 0271-4132; 397)
Includes bibliographical references.
ISBN 0-8218-3855-5 (alk. paper)
1. Riemann surfaces–Congresses. 2. Abelian groups–Congresses. I. Title: Geometry of Riemann surfaces and Abelian varieties. II. Muñoz Porras, Jose M. (José María), 1956- III. Popescu, Sorin, 1963- IV. Rodríguez, Rubí E., 1953- V. Recillas-Pishmish, Sevín, 1943- VI. Title. VII. Contemporary mathematics (American Mathematical Society); v. 397.

QA333.I24 2004
515′.93–dc22 2005057090

---



∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 11 10 09 08 07 06

# The Geometry of Riemann Surfaces and Abelian Varieties

# Titles in This Series

399 **Dominique Arlettaz and Kathryn Hess, Editors,** An Alpine anthology of homotopy theory, 2006

398 **Jay Jorgenson and Lynne Walling, Editors,** The ubiquitous heat kernel, 2006

397 **José M. Muñoz Porras, Sorin Popescu, and Rubí E. Rodríguez, Editors,** The geometry of Riemann surfaces and Abelian varieties, 2006

396 **Robert L. Devaney and Linda Keen, Editors,** Complex dynamics: Twenty-five years after the appearance of the Mandelbrot set, 2006

395 **Gary R. Jensen and Steven G. Krantz, Editors,** 150 Years of Mathematics at Washington University in St. Louis, 2006

394 **Rostislav Grigorchuk, Michael Mihalik, Mark Sapir, and Zoran Šunić, Editors,** Topological and asymptotic aspects of group theory, 2006

393 **Alec L. Matheson, Michael I. Stessin, and Richard M. Timoney, Editors,** Recent advances in operator-related function theory, 2006

392 **Stephen Berman, Brian Parshall, Leonard Scott, and Weiqiang Wang, Editors,** Infinite-dimensional aspects of representation theory and applications, 2005

391 **Jürgen Fuchs, Jouko Mickelsson, Grigori Rozenblioum, Alexander Stolin, and Anders Westerberg, Editors,** Noncommutative geometry and representation theory in mathematical physics, 2005

390 **Sudhir Ghorpade, Hema Srinivasan, and Jugal Verma, Editors,** Commutative algebra and algebraic geometry, 2005

389 **James Eells, Etienne Ghys, Mikhail Lyubich, Jacob Palis, and José Seade, Editors,** Geometry and dynamics, 2005

388 **Ravi Vakil, Editor,** Snowbird lectures in algebraic geometry, 2005

387 **Michael Entov, Yehuda Pinchover, and Michah Sageev, Editors,** Geometry, spectral theory, groups, and dynamics, 2005

386 **Yasuyuki Kachi, S. B. Mulay, and Pavlos Tzermias, Editors,** Recent progress in arithmetic and algebraic geometry, 2005

385 **Sergiy Kolyada, Yuri Manin, and Thomas Ward, Editors,** Algebraic and topological dynamics, 2005

384 **B. Diarra, A. Escassut, A. K. Katsaras, and L. Narici, Editors,** Ultrametric functional analysis, 2005

383 **Z.-C. Shi, Z. Chen, T. Tang, and D. Yu, Editors,** Recent advances in adaptive computation, 2005

382 **Mark Agranovsky, Lavi Karp, and David Shoikhet, Editors,** Complex analysis and dynamical systems II, 2005

381 **David Evans, Jeffrey J. Holt, Chris Jones, Karen Klintworth, Brian Parshall, Olivier Pfister, and Harold N. Ward, Editors,** Coding theory and quantum computing, 2005

380 **Andreas Blass and Yi Zhang, Editors,** Logic and its applications, 2005

379 **Dominic P. Clemence and Guoqing Tang, Editors,** Mathematical studies in nonlinear wave propagation, 2005

378 **Alexandre V. Borovik, Editor,** Groups, languages, algorithms, 2005

377 **G. L. Litvinov and V. P. Maslov, Editors,** Idempotent mathematics and mathematical physics, 2005

376 **José A. de la Peña, Ernesto Vallejo, and Natig Atakishiyev, Editors,** Algebraic structures and their representations, 2005

375 **Joseph Lipman, Suresh Nayak, and Pramathanath Sastry,** Variance and duality for cousin complexes on formal schemes, 2005

TITLES IN THIS SERIES

374 **Alexander Barvinok, Matthias Beck, Christian Haase, Bruce Reznick, and Volkmar Welker, Editors,** Integer points in polyhedra—geometry, number theory, algebra, optimization, 2005

373 **O. Costin, M. D. Kruskal, and A. Macintyre, Editors,** Analyzable functions and applications, 2005

372 **José Burillo, Sean Cleary, Murray Elder, Jennifer Taback, and Enric Ventura, Editors,** Geometric methods in group theory, 2005

371 **Gui-Qiang Chen, George Gasper, and Joseph Jerome, Editors,** Nonlinear partial differential equations and related analysis, 2005

370 **Pietro Poggi-Corradini, Editor,** The $p$-harmonic equation and recent advances in analysis, 2005

369 **Jaime Gutierrez, Vladimir Shpilrain, and Jie-Tai Yu, Editors,** Affine algebraic geometry, 2005

368 **Sagun Chanillo, Paulo D. Cordaro, Nicholas Hanges, Jorge Hounie, and Abdelhamid Meziani, Editors,** Geometric analysis of PDE and several complex variables, 2005

367 **Shu-Cheng Chang, Bennett Chow, Sun-Chin Chu, and Chang-Shou Lin, Editors,** Geometric evolution equations, 2005

366 **Bernhelm Booß-Bavnbek, Gerd Grubb, and Krzysztof P. Wojciechowski, Editors,** Spectral geometry of manifolds with boundary and decompositon of manifolds, 2005

365 **Robert S. Doran and Richard V. Kadison, Editors,** Operator algebras, quantization, and non-commutative geometry, 2004

364 **Mark Agranovsky, Lavi Karp, David Shoikhet, and Lawrence Zalcman, Editors,** Complex analysis and dynamical systems, 2004

363 **Anthony To-Ming Lau and Volker Runde, Editors,** Banach algebras and their applications, 2004

362 **Carlos Concha, Raul Manasevich, Gunther Uhlmann, and Michael S. Vogelius, Editors,** Partial differential equations and inverse problems, 2004

361 **Ali Enayat and Roman Kossak, Editors,** Nonstandard models of arithmetic and set theory, 2004

360 **Alexei G. Myasnikov and Vladimir Shpilrain, Editors,** Group theory, statistics, and cryptography, 2004

359 **S. Dostoglou and P. Ehrlich, Editors,** Advances in differential geometry and general relativity, 2004

358 **David Burns, Christian Popescu, Jonathan Sands, and David Solomon, Editors,** Stark's Conjectures: Recent work and new directions, 2004

357 **John Neuberger, Editor,** Variational methods: open problems, recent progress, and numerical algorithms, 2004

356 **Idris Assani, Editor,** Chapel Hill ergodic theory workshops, 2004

355 **William Abikoff and Andrew Haas, Editors,** In the tradition of Ahlfors and Bers, III, 2004

354 **Terence Gaffney and Maria Aparecida Soares Ruas, Editors,** Real and complex singularities, 2004

353 **M. C. Carvalho and J. F. Rodrigues, Editors,** Recent advances in the theory and applications of mass transport, 2004

352 **Marek Kubale, Editor,** Graph colorings, 2004

For a complete list of titles in this series, visit the AMS Bookstore at **www.ams.org/bookstore/**.

To our friend Sevín

# Preface

In September 1994, during an Algebraic Geometry Workshop in Morelia, Mexico, a number of Iberoamerican mathematicians (from Salamanca in Spain, Santiago and Valparaíso in Chile, and Morelia, Guanajuato and Mexico City in Mexico) who were working on similar types of problems joined their efforts and decided to meet again soon.

As a result, in July 1996, a workshop on "Abelian Varieties and Theta functions" was held in Morelia and its proceedings were published in Aportaciones Matemáticas, Serie Investigación **13**, Soc. Mat. Mex. 1998. The strong interest shown by participants motivated and fueled the idea to create a periodic mathematical framework to communicate, discuss and interact; and it was proposed that the next meeting should be held in Chile.

This idea was presented (and adhered to enthusiastically) at the Kra Fest at Stony Brook in 1997, and a multinational alliance (Chile, Mexico, Spain, USA) was formed to carry it out.

The "I Iberoamerican Congress on Geometry" was held in Olmué, Chile in January 1998, and its main topic was: The Geometry of Groups – Curves, Abelian Varieties, Theoretical and Computational Aspects. The congress proceedings were published in Contemp. Math. **240**, AMS 1999. The preface to that volume expressed the wish for this event to have been a continuation as well as a prelude to future Iberoamerican meetings.

This hope materialized as the "II Iberoamerican Congress on Geometry" was held at CIMAT in Guanajuato, Mexico, in January 2001. The main themes there were Complex Manifolds and Hyperbolic Geometry, and a proceedings was published in the current series: Contemp. Math. **311**, AMS. 2002. The participants at the congress agreed that the next meeting would be held in Spain.

As a result the third edition of the Iberoamerican Congress on Geometry was held at the University of Salamanca, Spain, in June 2004. The main themes here were Algebraic Curves, Riemann Surfaces, Modular Forms and Hyperbolic Geometry. It was an extremely successful event which provided the Iberoamerican community of geometers a wonderful opportunity to communicate and discuss their recent research.

The Salamanca meeting was dedicated to celebrate the $60^{\text{th}}$ birthday of our friend Sevín Recillas. He was a big creator of mathematics, friendships and research groups.

There are some mathematicians who prefer to work from behind the scenes. Sevín was of that kind, but just by looking a little deeper and asking a senior colleague or a young mathematician, one can find the deepest gratitude to him: a reflection on the beauty of mathematics, a loaned car, a big favor done, a good reference letter, some encouraging words at a critical moment ... These are some of the many different ways that he found to help, encourage, support and push forward those around him. Often these kinds of people are recognized after passing away. We were indeed fortunate to have given it to him before it was too late.

We would like to acknowledge the Institutions that financially supported the meeting and the participants, and thus made this congress possible. Funding came from Spain (University of Salamanca, DGI grant BFM2002-12288-E, Junta de Castilla y León and Caja Duero), United States of America (NSF grant DMS-0342699) and Chile (Fondecyt grant 1030595).

Our appreciation goes also to Ms. Christine M. Thivierge, Editorial Assistant to the Contemporary Mathematics series of AMS, and to Professors Esteban Gómez-González and Francisco Plaza-Martín from Salamanca, all of whom have greatly contributed to the publication of this proceedings.

After the congress, we have agreed to celebrate the next meeting in Ouro Preto, Brasil, in August 2007. We hope that the fourth event will turn out to be equally successful!

José M. Muñoz Porras
Sorin Popescu
Rubí E. Rodríguez

# Contents

Contemporary Mathematics
Volume **397**, 2006

# Möbius Transformations of the Circle Form a Maximal Convergence Group

Ara Basmajian and Mahmoud Zeinalian

Abstract. We investigate the relationship between quasisymmetric and convergence groups acting on the circle. We show that the Möbius transformations of the circle form a maximal convergence group. This completes the characterization of the Möbius group as a maximal convergence group acting on the sphere. Previously, Gehring and Martin had shown the maximality of the Möbius group on spheres of dimension greater than one. Maximality of the isometry (conformal) group of the hyperbolic plane as a uniform quasi-isometry group, uniformly quasiconformal group, and as a convergence group in which each element is topologically conjugate to an isometry may be viewed as consequences.

## 1. Introduction

The isometries of real hyperbolic space of dimension two or higher induce conformal diffeomorphisms on its ideal boundary. In fact, if the dimension is strictly greater than two, then all conformal diffeomorphisms will arise in this way. In contrast, in dimension two, conformality on the boundary is a trivial condition. For instance, every diffeomorphism of a Riemannian circle preserves the conformal class of the metric. It is for this reason that the study of the group of Möbius transformations of the circle differs from its higher dimensional cousins. In the paper [**Ge-M**], Gehring and Martin show that the Möbius group is a maximal convergence group acting on the boundary of real hyperbolic space of dimension greater than two. This result was extended (see [**B-Z**]) to the action of the isometry group of a rank one symmetric space of noncompact type except the hyperbolic plane.

In this note, we complete the characterization of the Möbius group as a maximal convergence group by considering the remaining case of the hyperbolic plane; namely, the group of Möbius transformations of the circle acts as a maximal convergence group (Theorem 3.3). Other maximality statements, such as the maximality of the isometry (conformal) group of the hyperbolic plane as a uniform quasi-isometry group and a uniformly quasiconformal group (see Corollary 4.2 and the discussion at the end of that section) may be regarded as consequences of Theorem 3.3. See [**Gr-P**] for further discussion on quasi-isometry groups. Another implication is the maximality of the isometry group of the hyperbolic plane as a

2000 *Mathematics Subject Classification.* Primary 30C62, Secondary 30F40.
*Key words and phrases.* convergence group, quasisymmetric, Möbius.

convergence group in which each element is topologically conjugate to an isometry (Corollary 4.3).

Let $X$ be a compact topological space. A family $\mathcal{F}$ of orientation preserving homeomorphisms of $X$ is said to have the *convergence property* if each infinite sequence $\{f_n\}$ of $\mathcal{F}$ contains a subsequence which,

- **C1:** converges uniformly to a homeomorphism of $X$, or
- **C2:** has the attractor-repeller property, that is, there exists a point $a \in X$, the *attractor*, and a point $r \in X$, the *repeller*, so that the $\{f_n\}$ converge to the constant function $a$, uniformly outside of any open neighborhood of $r$. Note that $a$ may equal $r$.

We remark that the convergence groups considered in this paper are comprised only of orientation preserving homeomorphisms. We could equally as well include orientation reversing homeomorphisms, in which case the theorems in this paper have obvious modifications that are left to the reader. Hence, whenever homeomorphism is mentioned in this paper it is assumed to be orientation preserving.

## 2. Elementary facts about quasisymmetric mappings

In this section, we assemble some elementary facts which will be needed later in the paper. For the basics on quasisymmetric and quasiconformal maps, we refer to the following papers and books: [**A**], [**D-E**], [**Ga-L**], [**H**], [**L**], and [**V**]. For Möbius groups and hyperbolic geometry, the reader may consult [**Be**] or [**M**].

Let $\mathbb{H}$ denote the upper half plane, endowed with the hyperbolic metric, and $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ denote its ideal boundary. $\widehat{\mathbb{R}}$ can be identified with the unit circle $S^1$. The group $\mathrm{PSL}(2,\mathbb{R}) = \{z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R}, ad-bc=1\}$ is the full group of orientation preserving isometries of $\mathbb{H}$. This group is also the full group of conformal homeomorphisms of $\mathbb{H}$. Let $\text{Möb}^+(\widehat{\mathbb{R}})$ denote the group of homeomorphisms of $\widehat{\mathbb{R}}$ which are induced by the isometries of $\mathbb{H}$. Note that $\mathrm{PSL}(2,\mathbb{R})$ and $\text{Möb}^+(\widehat{\mathbb{R}})$ are isomorphic groups which act on different spaces.

When dealing with mappings of the circle we will need to normalize by post composition using an element of $\text{Möb}^+(\widehat{\mathbb{R}})$ so that infinity is a fixed point of the map. Observe that an element of the stabilizer of $\infty$ in $\text{Möb}^+(\widehat{\mathbb{R}})$ is a linear map, $x \mapsto mx + b$, where $m > 0$ and $b \in \mathbb{R}$. Hence given a symmetric configuration of point triples, $\{x-t, x, x+t\}$, its image remains a symmetric configuration.

Definition 2.1. Let $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ be an orientation preserving homeomorphism and $k > 0$. The homeomorphism $f$ is called $k$-quasisymmetric if after normalizing so that it fixes infinity, $f$ satisfies,

$$\frac{1}{k} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq k$$

for all $x \in \mathbb{R}$ and all $t > 0$. In other words, the image of equal length juxtaposed intervals has uniform bounded length ratio.

Given the observation preceding Definition 2.1, it is easy to see that the condition of being $k$-quasisymmetric is independent of the normalizing Möbius transformation, the elements of $\text{Möb}^+(\widehat{\mathbb{R}})$ are 1-quasisymmetric, and that post or precomposition by linear maps does not change the quasisymmetric constant.

In the sequel, we will need the fact that $\text{Möb}^+(\widehat{\mathbb{R}})$ is the full group of 1-quasisymmetric homeomorphisms. To see this, using the triple transitivity of

$\text{Möb}^+(\widehat{\mathbb{R}})$, it is enough to show that a 1-quasisymmetric mapping $f$ which fixes $0, 1$, and $\infty$ is the identity. Now, the triple $\{0, \frac{1}{2}, 1\}$ must be taken to a symmetric triple, hence $f$ also fixes $\frac{1}{2}$. Similar considerations allow us to conclude that all rationals of the form $\frac{n}{2^k}$ are fixed. Finally, using continuity, $f$ must fix every real number and thus is the identity homeomorphism.

The following proposition is a classical result. See Ahlfors [**A**] or Lehto [**L**] for reference.

PROPOSITION 2.2. *Let $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ be an increasing homeomorphism with $f(\infty) = \infty$. Assume that*

$$\frac{1}{k} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq k$$

*for all $x$ and all $t > 0$. Then there exists a $K(k)$ -quasiconformal homeomorphism of $\mathbb{H}$ which extends $f$. The number $K(k)$ depends only on $k$.*

## 3. Maximality of $\text{Möb}^+(\widehat{\mathbb{R}})$

A family $\mathcal{F}$ is said to be *uniformly quasisymmetric (quasiconformal)* if all the maps in $\mathcal{F}$ are $k$-quasisymmetric ($K$-quasiconformal) for some $k$ (for some $K$).

PROPOSITION 3.1. *A uniformly quasisymmetric family $\mathcal{F}$ of homeomorphisms of $\widehat{\mathbb{R}}$ is a convergence family. In particular, $\text{Möb}^+(\widehat{\mathbb{R}})$ is a convergence group.*

PROOF. Using Proposition 2.2, there exists a number $K$ such that every element of this family can be extended to a $K$-quasiconformal mapping of $\mathbb{H}$. Using the fact that a sequence of distinct $K$-quasiconformal mappings of $\mathbb{H}$ has a subsequence which either converges to a $K$-quasiconformal map or has the attractor-repeller property with attractor and repeller on the boundary (see [**V**], Corollaries 19.3 and 37.4, or extend each map in $\mathcal{F}$ to $S^2$ by reflection and use the results of [**Ge-M**]), we may conclude it acts as a convergence family on $S^1$. □

PROPOSITION 3.2. *Let $\mathcal{F}$ be a family of homeomorphisms of $\widehat{\mathbb{R}}$ which is closed under post and precomposition by elements of $\text{Möb}^+(\widehat{\mathbb{R}})$. Then $\mathcal{F}$ has the convergence property if and only if $\mathcal{F}$ is a uniformly quasisymmetric family.*

PROOF. Suppose $\mathcal{F}$ has the convergence property. Let

$$\mathcal{F}' = \{f \in \mathcal{F} : f(0) = 0, f(1) = 1, \text{ and } f(\infty) = \infty\}. \tag{1}$$

Since $\mathcal{F}'$ has the convergence property, it must be that there are negative constants $M$ and $m$ so that, $m < f(-1) < M < 0$, for all $f \in \mathcal{F}'$. Any element of $\mathcal{F}$ can be post composed by an element of $\text{Möb}^+(\widehat{\mathbb{R}})$ to yield an element of $\mathcal{F}'$. Since any triple $\{x-t, x, x+t\}$ in $\mathbb{R}$ can be moved by Euclidean translation and dilation to $\{-1, 0, 1\}$, we may conclude that the elements of $\mathcal{F}$ form a uniformly quasisymmetric family. The converse follows from Proposition 3.1.

□

THEOREM 3.3. *$\text{Möb}^+(\widehat{\mathbb{R}})$ acts on $\widehat{\mathbb{R}}$ as a maximal convergence group. That is, there is no convergence group that properly contains $\text{Möb}^+(\widehat{\mathbb{R}})$.*

PROOF. The fact that $\text{Möb}^+(\widehat{\mathbb{R}})$ is a convergence group follows from Proposition 3.1. Next let $G$ be a convergence group acting on $\widehat{\mathbb{R}}$ containing $\text{Möb}^+(\widehat{\mathbb{R}})$.

Using Proposition 3.2, we know that the action of $G$ is as a uniformly quasisymmetric group. On the other hand, suppose there exists an element $g \in G$ not contained in $\text{Möb}^+(\widehat{\mathbb{R}})$. This means that after normalizing $g$, so that it fixes $\infty$, there must be three symmetrically spaced points in $\mathbb{R}$ where the quasisymmetric constant is not 1. Post and precomposing by Euclidean translations and dilations, we may assume that the three points are $\{-1, 0, 1\}$ and that $g$ fixes 0 and 1. Since Euclidean translation and dilation do not effect the quasisymmetric constant for a triple, it must be that $g$ does not fix $-1$. By possibly replacing $g$ with $g^{-1}$, we may assume that $g$ takes $-1$ into the interval $(-1, 0)$. Clearly $g^n(-1)$ is an increasing sequence of negative numbers and hence has a limit $y$ which is necessarily a fixed point of $g$. Since $< g >$ is a convergence group, $y$ is strictly less than 0. Next consider the triple of points $\{-1, y, 0\}$. The length ratio of the juxtaposed intervals $[-1, y]$ and $[y, 0]$ is,

$$\frac{|0 - y|}{|y - (-1)|} = \frac{|y|}{|y + 1|}. \tag{2}$$

The length ratio of image intervals under the iterates of $g$ are

$$\frac{|g^n(0) - g^n(y)|}{|g^n(y) - g^n(-1)|} = \frac{|y|}{|y - g^n(-1)|} \tag{3}$$

which goes to $\infty$, as $n \to \infty$. This contradicts the fact that $G$ is uniformly quasisymmetric. Hence, it must be that the quasisymmetric constant for $g$ is 1, and thus $g \in \text{Möb}^+(\widehat{\mathbb{R}})$. □

## 4. Maximality of PSL$(2, \mathbb{R})$

An immediate corollary of Theorem 3.3 is,

COROLLARY 4.1. *The Möbius group, $\text{Möb}^+(\widehat{\mathbb{R}})$, is a maximal uniformly quasisymmetric group.*

A homeomorphism $f : \mathbb{H} \to \mathbb{H}$ is said to be a *quasi-isometry* if there exist positive constants $A$ and $B$ so that

$$A^{-1}d(x_1, x_2) - B \le d(f(x_1), f(x_2)) \le Ad(x_1, x_2) + B \tag{4}$$

for all $x_1, x_2 \in \mathbb{H}$. The constant $A$ is referred to as the *Lipschitz constant* of the quasi-isometry. For a general reference on quasi-isometries, we refer the reader to [**Gr-P**]. It is well known that a quasi-isometry continuously extends to the boundary and that the induced map on the boundary is a quasisymmetric homeomorphism. One defines an equivalence relation on quasi-isometries by declaring two to be equivalent if they induce the same homeomorphism on the boundary. Let QI$(\mathbb{H})$ denote the group of equivalence classes of quasi-isometries of $\mathbb{H}$. Since the natural map from PSL$(2, \mathbb{R})$ into QI$(\mathbb{H})$ is injective, we will continue to denote its image with the same notation. A family $\mathcal{F} \subset$ QI$(\mathbb{H})$ is said to be a *uniformly quasi-isometric* family if each equivalence class has Lipschitz constant less than a uniform bound. The following is a simple consequence of Theorem 3.3, observed in Gromov and Pansu (see [**Gr-P**]).

COROLLARY 4.2. *Let $G \le QI(\mathbb{H})$ be a uniform quasi-isometry group acting on the hyperbolic plane $\mathbb{H}$. If $PSL(2, \mathbb{R}) \le G$, then $G = PSL(2, \mathbb{R})$.*

PROOF. A quasi-isometry $f$ of $\mathbb{H}$ extends to a homeomorphism of $\mathbb{H}\cup\widehat{\mathbb{R}}$. Moreover, the induced mapping $f|_{\widehat{\mathbb{R}}}$ on $\widehat{\mathbb{R}}$ is a quasisymmetric mapping where the quasisymmetric constant depends only on the Lipschitz constant of the quasi-isometry. Consider the homomorphism $\phi : G \to Homeo(\widehat{\mathbb{R}})$, given by $[f] \mapsto f|_{\widehat{\mathbb{R}}}$. Note that Image($\phi$) is a uniformly quasisymmetric group which contains Möb$^+(\widehat{\mathbb{R}})$. By Proposition 3.1, Image($\phi$) is a convergence group. Since Möb$^+(\widehat{\mathbb{R}})$ is a maximal convergence group (Theorem 3.3), Image($\phi$) equals Möb$^+(\widehat{\mathbb{R}})$. Injectivity of $\phi$ is a tautology. Since Image($\phi$) $=$ $\phi(\mathrm{PSL}(2,\mathbb{R}))$ $=$ Möb$^+(\widehat{\mathbb{R}})$, we conclude that $G = \mathrm{PSL}(2,\mathbb{R})$. □

As in the proof above, it is easy to see that a convergence group acting on $\mathbb{H}\cup\widehat{\mathbb{R}}$ which contains PSL$(2,\mathbb{R})$ induces the action of Möb$^+(\widehat{\mathbb{R}})$ on the boundary, $\widehat{\mathbb{R}}$. Furthermore, if each element of this convergence group is topologically conjugate to an element of PSL$(2,\mathbb{R})$, then the induced action has trivial kernel. Hence the natural homomorphism given by restriction of the convergence group to the boundary is in fact an isomorphism onto Möb$^+(\widehat{\mathbb{R}})$. Since the image of PSL$(2,\mathbb{R})$ is Möb$^+(\widehat{\mathbb{R}})$, it must be that the convergence group equals PSL$(2,\mathbb{R})$. We have proven,

COROLLARY 4.3. *Let $G$ be a convergence group acting on $\mathbb{H}\cup\widehat{\mathbb{R}}$. Suppose that every element of $G$ is topologically conjugate to an element of PSL$(2,\mathbb{R})$. If PSL$(2,\mathbb{R}) \leq G$, then $G =$ PSL$(2,\mathbb{R})$. The conjugating homeomorphism need not be the same for all elements of $G$.*

The reader should compare the above corollary to the fact that PSL$(2,\mathbb{R})$ is a maximal uniformly quasiconformal group acting on $\mathbb{H}$. That is, if $G$ is a uniformly quasiconformal group containing PSL$(2,\mathbb{R})$, then $G =$ PSL$(2,\mathbb{R})$. This fact follows from the results of [**S**] and [**T**].

## References


[A] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand-Reinhold, Princeton, New Jersey, (1966).

[Be] Alan F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1983. xii+337 pp.

[B-Z] A. Basmajian, M. Zeinalian, *Maximal convergence groups and rank one symmetric spaces*, preprint.

[D-E] A. Douady, C. J. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. 157 no. 1-2 (1986), 23–48.

[Ga-L] F. P. Gardiner, N. Lakic, *Quasiconformal Teichmuller Theory*, American Mathematical Society, Mathematical Surveys and Monographs, 76 (2000).

[Ge-M] F. W. Gehring, G. J. Martin, *Discrete quasiconformal groups I* , Proc. London Math. Soc. (3), Vol 55, (1987) 331–358.

[Gr-P] M. Gromov, P. Pansu, *Rigidity of lattices: An introduction*, Geometric Topology: Recent developments, Lecture Notes in Math. 1504, Springer, Berlin (1991), 39–137.

[H] A. Hinkkanen, *Uniformly quasisymmetric groups.* Proc. London Math. Soc. (3) 51 (1985), no. 2, 318–338.

[L] O. Lehto, *Univalent Functions and Teichmülar Spaces*, Springer-Verlag, (1987).

[M] B. Maskit, *Kleinian groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 287. Springer-Verlag, Berlin, 1988. xiv+326 pp.

[S] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (1978), 465–496; Ann. of Math. Stud., 97, Princeton Univ. Press (1981).

[T] P. Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. 5 no. 1 (1980), 73–78.

[V] J. Vaisala, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. 229, Springer-Verlag, Berlin-New York (1971).

Ara Basmajian, Department of Mathematics, University of Oklahoma, Norman, OK 73019, abasmajian@ou.edu

Mahmoud Zeinalian, Department of Mathematics, C.W.Post Campus, Long Island University, 720 Northern Boulevard, Brookville, NY 11548, mzeinalian@liu.edu

Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019
*E-mail address*: `abasmajian@ou.edu`

Department of Mathematics, C.W.Post College of Long Island University, 720 Northern Boulevard, Brookville, New York 11548
*E-mail address*: `mzeinalian@liu.edu`