LINEAR ANALYSIS

MEASURE AND INTEGRAL, BANACH AND HILBERT SPACE, LINEAR INTEGRAL EQUATIONS

BY

ADRIAAN CORNELIS ZAANEN
PROFESSOR OF MATHEMATICS
AT THE TECHNOLOGICAL UNIVERSITY
DELFT, NETHERLANDS

1953

PREFACE

The present work is devoted to some mathematical theories which all of them may claim to belong to the domain of "linear analysis". It will, however, be clear that it is impossible to comprise everything that might deserve to be so designated into a book of moderate size. The central and larger part of the book (Ch. 6—12) is formed by an introduction to Banach space and Hilbert space, and to some aspects of the theory of bounded and compact linear transformations in these spaces. In order to have at hand non-trivial examples by which to illustrate this theory, the necessity arises to make an appeal to such notions as Lebesgue measure and integral, and Lebesgue function spaces L_p. It is to a concise discussion of these subjects, complete in itself but leaving aside everything which is unessential for the applications in question, that Part I (Ch. 1—5) is devoted. The emphasis is on the algebraic aspects of measure and integral, and topological notions are only introduced when discussing special examples such as Lebesgue or Stieltjes integrals. I wish to express my gratitude to Prof. N. G. de Bruijn for the free use I could make of his lecture notes on measure and integral. The study of these notes led to the dominant part played by semirings in Chapter 2, and to the definition of the integral as a product measure in the product space of the underlying measure space and the straight line. Furthermore it will appear on comparison with the standard treatises of Halmos (Measure Theory, New York, 1950) and Saks (Theory of the Integral, Warsaw, 1937) that they have not failed to leave their marks in the present discussion. In Chapter 5 we introduce besides the Lebesgue spaces $L_p(1 \leqslant p \leqslant \infty)$, their generalizations: the Orlicz spaces L_Φ. Most of the properties of spaces L_p may be carried over to spaces L_Φ, although several proofs need some modification. As a rule the proofs in question gain thereby in transparency, since a certain amount of juggling with conjugated exponents p and q is replaced by more straightforward arguments.

After defining Banach and Hilbert space in Chapter 6, and deriving their most obvious properties, Chapter 7 is devoted to a first study of bounded linear transformations in Banach space, although closed, possibly unbounded, linear transformations are not altogether forgotten. Bounded linear functionals, their extension, the first and second adjoint transformations, weak convergence and projections are among the subjects which receive attention. Chapter 8 deals with linear transformations in a space of finite dimension (rank and nullity, characteristic values, canonical form of a matrix, elementary divisors), and Chapter 9 with bounded linear transformations in Hilbert space (unitary, self-adjoint, normal and symmetrisable transformations, projections, Fredholm theory for transformations of finite double-norm). The spectral theory of bounded normal transformations has not been included, since there exist several treatises containing an excellent discussion of this subject (cf. the references in Ch. 6, § 1). In Chapter 10 the mutual relations between the ranges and null spaces of a bounded linear transformation and its adjoint are discussed, and we introduce resolvent and spectrum, while Chapter 11 gives a rather thorough account of compact linear transformations in Banach space (Riesz-Schauder theory, resolvent, algebraic multiplicity of characteristic values, mean ergodic theorems). Finally, in Chapter 12, the last chapter of Part II, we consider compact transformations in Hilbert space which are either self-adjoint, normal or symmetrisable (expansion theorems, minimax theorems, perturbation theorems). The reader who is not interested in symmetrisable transformations, will find some advice on how to read this chapter in § 1.

Examples illustrating the abstract theory, are scattered through the text, and more of them may be found at the end of each chapter.

Part III (Ch. 13—17) deals with non-singular linear integral equations, that is, with the Fredholm theory and spectral theory of linear integral transformations one of whose iterates is compact. In Chapter 13, which may be read, if one wishes, immediately after Chapter 11, the Fredholm theory is lifted out of the L_2-sphere, wherein it was imprisoned until rather recently, and placed in more natural surroundings: the Lebesgue spaces L_p and the Orlicz spaces L_Φ. In Chapter 14 equations with normal, Hermitian or positive definite kernels are discussed, and the remaining chapters, finally, deal with equations having a symmetrisable kernel. Anyone who has ever taken note of what Hellinger and Toeplitz in their "Encyklopädie" paper of 1927 say about equations with symmetrisable kernels, will know that the state of affairs at that time was rather unsatisfactory. We cite: "Der wesentliche Mangel dieses ganzen allgemeinen Ansatzes ist aber das Fehlen der eigentlichen Entwicklungssätze", and "... der Ansatz der symmetrisierbaren Kerne in seiner formalen Allgemeinheit ins Unbestimmte greift". Fortunately, it has been proved since 1927 that the situation is not so bad as one might infer from these quotations. There exist expansion theorems in the case of a general symmetrisable kernel (Marty kernel), although of a slightly different nature than in the case where we have to do with a Hermitian kernel.

The knowledge required for a proper understanding of the contents of the present work, does nowhere, with one exception, go beyond the elements of ordinary analysis (determinants and linear equations, continuity, Riemann integral, ordinary and uniform convergence of series, complex numbers, simple properties of power series). The exception is in Ch. 13, § 14, where we use a theorem on entire functions which lies somewhat deeper.

The conventions on cross references are as follows: by § 4, Th. 2 is meant § 4, Theorem 2 of the chapter in which the reference occurs, and by Theorem 3 is meant Theorem 3 of the paragraph in which the reference occurs. Numerals in square brackets refer to the list of bibliographical references at the end of each chapter. No claim to completeness is made with regard to these lists. The idea is merely to illustrate some striking points in the text by a name and a date. Part I, being of an introductory nature, contains no bibliographical references.

All suggestions from readers which might lead to future improvements in the text will be very welcome.

My sincere thanks are due to the editors of the "Bibliotheca Mathematica" for their invitation to publish this book in their series. Furthermore, I wish to express my gratitude to Prof. H. D. Kloosterman who critically examined part of the manuscript, and to W. A. J. Luxemburg who assisted in the proof reading.

Delft, May 1953 A. C. ZAANEN

PART I

MEASURE AND INTEGRAL, THE LEBESGUE FUNCTIONSPACES L_p AND THE ORLICZ FUNCTIONSPACES L_Φ

CHAPTER 1

POINT SETS. EUCLIDEAN SPACE

§ 1. Definitions and Some Simple Properties

We consider a set X, the elements $x, y, \ldots$ of which will be called points. If A is a subset of X, we write $x \in A$ in case the point x is an element of A. Hence $x \in X$ for every point x. Given two subsets A and B of X, we write $A \subset B$ or, equivalently, $B \supset A$ when A is a subset of B, in other words, when every point of A is also a point of B. In particular $A \subset A$ for every set A. The *empty set*, that is, the set containing no points at all, is denoted by 0, and this set is considered to be a subset of every subset. Hence $0 \subset A$ for every $A \subset X$.

Given a sequence (finite or infinite) $A_n (n = 1, 2, \ldots)$ of sets (we shall not repeat every time that all point sets considered are subsets of X), we call *sum*, or *union*, of these sets and denote by ΣA_n, or $A_1 + A_2 + \ldots$, the set of all points belonging to at least one of the A_n. We call *product*, or *intersection*, and denote by ΠA_n, or $A_1 A_2 \ldots$, the set of all points belonging at the same time to all the A_n. Given two sets A and B, the set of all points belonging to A, but not to B, is called the *difference* of A and B, and denoted by $A - B$. By $A - B - C$ we mean $(A - B) - C$.

If the sequence A_n is infinite, the set of all points x such that $x \in A_n$ holds for an infinity of values of n is called the *upper limit* of A_n, and denoted by lim sup A_n. The set of all points x belonging to all the sets A_n from some n_0 onwards (n_0 may vary with x) is called the *lower limit* of the sequence and denoted by lim inf A_n. Obviously lim inf $A_n \subset$ lim sup A_n. Whenever lim inf A_n = lim sup A_n ($A = B$ for two sets A and B means that $A \subset B$ and $B \subset A$ hold simultaneously), the sequence A_n is said to be convergent; lower and upper limit are called *limit*, and denoted by lim A_n.

Theorem 1. *lim sup* $A_n = \Pi_{k=1}^{\infty} \Sigma_{n=k}^{\infty} A_n$.

Proof. Let $x \in \limsup A_n$. Then $x \in \Sigma_{n=k}^{\infty} A_n$ for every k, hence $x \in \Pi_{k=1}^{\infty} \Sigma_{n=k}^{\infty} A_n$.

Let conversely $x \in \Pi_{k=1}^{\infty} \Sigma_{n=k}^{\infty} A_n$. Then $x \in \Sigma_{n=k}^{\infty} A_n$ for every k, which shows that $x \in A_n$ holds for an infinity of values of n. Hence $x \in \lim\sup A_n$.

Theorem 2. *lim inf* $A_n = \Sigma_{k=1}^{\infty} \Pi_{n=k}^{\infty} A_n$.

Proof. Let $x \in \lim\inf A_n$. Then there exists a positive integer k_0 such that $x \in A_n$ for $n \geqslant k_0$. Hence $x \in \Pi_{n=k_0}^{\infty} A_n$, which implies $x \in \Sigma_{k=1}^{\infty} \Pi_{n=k}^{\infty} A_n$.

Let conversely $x \in \Sigma_{k=1}^{\infty} \Pi_{n=k}^{\infty} A_n$. Then $x \in \Pi_{n=k}^{\infty} A_n$ for at least one index k, hence $x \in A_n$ for $n \geqslant k$ or $x \in \lim\inf A_n$.

If, for a sequence A_n of sets, $A_n \subset A_{n+1}$ for all n, the sequence is called *ascending* or *non-decreasing*; if $A_{n+1} \subset A_n$ for all n, the sequence is called *descending* or *non-increasing*. Ascending and descending sequences are said to be *monotone*.

Theorem 3. 1°. *A monotone (infinite) sequence is convergent.*

2°. *lim* $A_n = \Sigma A_n$ *for an ascending sequence.*

3°. *lim* $A_n = \Pi A_n$ *for a descending sequence.*

Proof. 1°. We have to prove that $\lim\sup A_n \subset \lim\inf A_n$. Let us assume, for this purpose, that $x \in \lim\sup A_n$, so that $x \in A_n$ holds for an infinity of values of n. If now the sequence A_n is ascending and n_0 is the smallest index n for which $x \in A_n$ holds, then $x \in A_n$ for $n \geqslant n_0$, and hence $x \in \lim\inf A_n$. If, on the other hand, the sequence A_n is descending and $x \in A_k$ for a certain k, then $x \in A_n$ for $n \leqslant k$. Hence, since $x \in A_k$ holds for infinitely many k, we have $x \in A_n$ for all n, so that certainly $x \in \lim\inf A_n$.

2°. If the sequence A_n is ascending, we have

$$\lim A_n = \lim\inf A_n = \Sigma_{k=1}^{\infty} \Pi_{n=k}^{\infty} A_n = \Sigma_{k=1}^{\infty} A_k.$$

3°. If the sequence A_n is descending, we have

$$\lim A_n = \lim\sup A_n = \Pi_{k=1}^{\infty} \Sigma_{n=k}^{\infty} A_n = \Pi_{k=1}^{\infty} A_k.$$

If A is a subset of X, the set $X - A$ is called the *complementary set* or, shortly, the *complement* of A, and is sometimes denoted by A'. Evidently $X' = 0$, $0' = X$, $(A')' = A$, and $A \subset B$ implies $A' \supset B'$.

Theorem 4. *We have* $A - B = AB'$.

Proof. We have $x \in A - B$ if and only if $x \in A$ and $x \in B'$ hold simultaneously.

Theorem 5. *We have* $\Pi A_n = (\Sigma A_n')'$ *and* $\Sigma A_n = (\Pi A_n')'$.

Proof. We have $x \in \Pi A_n$ if and only if $x \in A_n'$ holds for no value of n, hence, if and only if $x \in (\Sigma A_n')'$.

We have $x \in \Sigma A_n$ if and only if $x \in A_n'$ does not hold for all values of n, hence, if and only if $x \in (\Pi A_n')'$.

Theorem 6. *We have* $\lim\sup A_n = (\lim\inf A_n')'$ *and* $\lim\inf A_n = (\lim\sup A_n')'$.

Proof. Similar to that of the preceding theorem.

Remark. The set $A - B$ is called the complement of B relative to A. It is easily seen that, if all sets A_n are contained in A, the Theorems 5 and 6 remain true also if all complements are taken relative to A (A takes over the part of X).

The function of a point, equal to one at all points of a set A, and to zero at all points of $A' = X - A$, is called the *characteristic function* $c_A(x)$ of A. Evidently, if $A = \Sigma A_n$ and $A_i A_j = 0$ $(i \neq j)$, that is, if no two of the sets A_n have common points, then $c_A(x) = \Sigma c_{A_n}(x)$. If the sequence A_n is monotone, the sequence of the characteristic functions is (for every point x) also monotone, non-decreasing if A_n is ascending and non-increasing if A_n is descending.

Theorem 7. *If* $\lim\sup A_n = P$ *and* $\lim\inf A_n = Q$, *then*

$$c_P(x) = \lim\sup c_{A_n}(x),$$

$$c_Q(x) = \lim\inf c_{A_n}(x).$$

Proof. We have $c_P(x) = 1$ if and only if $x \in A_n$ holds for infinitely many values of n, that is, if and only if $c_{A_n}(x) = 1$ for infinitely many values of n. Hence $c_P(x) = 1$ if and only if $\lim\sup c_{A_n}(x) = 1$. Since both $c_P(x)$ and $\lim\sup c_{A_n}(x)$ can only assume the values zero and one, the relation $c_P(x) = \lim\sup c_{A_n}(x)$ holds for all x.

We have $c_Q(x) = 1$ if and only if there exists an index n_0 such that $x \in A_n$ for $n \geqslant n_0$, that is, if and only if $c_{A_n}(x) = 1$ for $n \geqslant n_0$. Hence $c_Q(x) = 1$ if and only if $\lim\inf c_{A_n}(x) = 1$. Since both $c_Q(x)$ and $\lim\inf c_{A_n}(x)$ can only assume the values zero and one, there is equality for all x.

Theorem 8. *The sequence A_n converges to the set A if and only if*

$$\lim c_{A_n}(x) = c_A(x).$$

Proof. Let $A = \lim A_n$, so that, with the notations of the preceding theorem, $P = A = Q$. Then

$$\limsup c_{A_n}(x) = c_P(x) = c_A(x) = c_Q(x) = \liminf c_{A_n}(x),$$

hence $c_A(x) = \lim c_{A_n}(x)$.

Let now conversely $c_A(x) = \lim c_{A_n}(x) = \limsup c_{A_n}(x) = \liminf c_{A_n}(x)$, so that $c_A(x) = c_P(x) = c_Q(x)$. Then $A = P = Q$, hence $A = \lim A_n$.

A function $f(x)$, assuming only a finite number of different (real or complex) values on X is called a *simple function*. If these values are $\alpha_1, \ldots, \alpha_p$, and the set on which the value α_i is taken is denoted by $A_i (i = 1, \ldots, p)$, then $X = \Sigma_{i=1}^{p} A_i$, where no two of the sets A_i have common points, and $f(x) = \Sigma_{i=1}^{p} \alpha_i c_{A_i}(x)$.

§ 2. Euclidean Space

By *Euclidean space* of m dimensions R_m we mean the set of all systems of m real numbers $(x_1, x_2, \ldots, x_m)$. The number x_k is called the k-th *coordinate* of the point $x = (x_1, x_2, \ldots, x_m)$. Obviously all notions introduced in the preceding paragraph, may be applied to point sets in R_m. We shall introduce here some further conceptions in which the distance $\rho(x, y)$ between two points x and y of R_m plays a part.

The *distance* $\rho(x, y)$ between $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ is defined as the non-negative number

$$[(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_m - y_m)^2]^{1/2}.$$

The space R_1 is also termed *straight line* and the space R_2 *plane*. In R_1 the distance $\rho(x, y)$ is simply $| x - y |$.

If E is a subset of R_m, the upper bound (also called lowest upper bound; abbreviation l.u.b.) of the numbers $\rho(x, y)$ subject to $x \in E$ and $y \in E$ is the *diameter* of E, and is denoted by $\delta(E)$. If $\delta(E)$ is finite, the set E is called *bounded*. For a collection M of sets, the upper bound of $\delta(E)$ for all sets E belonging to M is called the *characteristic number* of M. By the distance $\rho(x, E)$ of a point x and a set E we mean the lower bound (often called greatest lower bound; abbreviation g.l.b.) of $\rho(x, y)$ for all $y \in E$, and by the distance $\rho(E_1, E_2)$ of two sets E_1 and E_2 the lower bound of $\rho(x, y)$ for x running through E_1 and y through E_2.

If $r > 0$ is given, the set of all points y such that $\rho(x, y) < r$ is called the (*spherical*) *neighbourhood* of x with radius r. A point x is called a *point of accumulation* of a set E when every neighbourhood of x contains infinitely many points of E. It is not necessary, therefore, that x itself belongs to E. This definition is equivalent to the definition that x is a point of accumulation of E when every neighbourhood of x contains at least one point of E different from x. All points of E that are no points of accumulation of E are said to be *isolated points* of E. The set E^+ of all points of accumulation of E is the *derived set* of E. If E^+ is contained in E, the set E is termed *closed*. For every set E, the set E^+ is closed. Indeed, let x be a point of accumulation of E^+. We have to prove that $x \in E^+$. If $r > 0$ is arbitrarily given, there exists a point $y \in E^+$ in the neighbourhood of x with radius $r/2$. Consider now a neighbourhood of y with radius smaller than $r/2$. This neighbourhood contains, on account of $y \in E^+$, an infinity of points of E. Since moreover it is wholly contained in the neighbourhood of x with radius r, we have proved that infinitely many points of E are lying in the latter neighbourhood, and this shows that $x \in E^+$.

A point x is called *limit* of a sequence of points x_n when $\lim \rho(x, x_n) = 0$. We write $x = \lim x_n$, and the sequence x_n is said to be converging to x. Obviously, by the definition of distance and by Cauchy's Theorem, the sequence x_n is convergent if and only if $\lim \rho(x_p, x_q) = 0$ as $p, q \to \infty$. Furthermore, it is evident that the limit of a convergent sequence with an infinity of different points belonging to a set E is a point of accumulation of E.

If two points $a = (a_1, a_2, \ldots, a_m)$ and $b = (b_1, b_2, \ldots, b_m)$ in R_m are given, and if $a_k < b_k$ for $k = 1, 2, \ldots, m$, we call *closed interval* $[a_1, b_1; \ldots; a_m, b_m]$ the set of all points $(x_1, x_2, \ldots, x_m)$ such that $a_k \leqslant x_k \leqslant b_k$ for $k = 1, 2, \ldots, m$. If we replace $a_k \leqslant x_k \leqslant b_k$ by $a_k < x_k < b_k$ or $a_k \leqslant x_k < b_k$ or $a_k < x_k \leqslant b_k$, we obtain the definition of *open interval* $(a_1, b_1; \ldots; a_m, b_m)$ or *interval right open* $[a_1, b_1; \ldots; a_m, b_m)$ or *interval left open* $(a_1, b_1; \ldots; a_m, b_m]$ respectively. In what follows we shall mean by interval a closed interval, whereas an interval left open will be called a *cell*.

Theorem 1 (*Theorem of Bolzano-Weierstrass*). *Every bounded set in R_m containing an infinity of points has at least one point of accumulation.*

Proof. If E is the set in question, it is contained in an interval V_1. Dividing V_1 into 2^m congruent non-overlapping intervals, such that the diameter of each of these intervals is $\delta(V_1)/2$, one at least of these intervals, say V_2, contains an infinity of points of E. Repeating this process, we obtain a descending sequence of intervals $V_n (n = 1, 2, \ldots)$ such that each V_n contains infinitely many points of E and $\lim \delta(V_n) = 0$. This implies that, if $V_n = [a_1^{(n)}, b_1^{(n)}; \ldots; a_m^{(n)}, b_m^{(n)}]$, the sequence of numbers $a_k^{(1)}, a_k^{(2)}, \ldots$ is non-decreasing, the sequence $b_k^{(1)}, b_k^{(2)}, \ldots$ is non-increasing and $\lim (b_k^{(n)} - a_k^{(n)}) = 0$ for $k = 1, \ldots, m$. We may conclude therefore that $x_k = \lim a_k^{(n)} = \lim b_k^{(n)}$ exists for $k = 1, \ldots, m$. The point $x = (x_1, x_2, \ldots, x_m)$ is now a point of accumulation of E. Indeed, since evidently $x \in V_n$ for all n and $\lim \delta(V_n) = 0$, the intervals V_n are contained in an arbitrary neighbourhood of x for n sufficiently large, so that an infinity of points of E is lying in this neighbourhood.

§ 3. Open and Closed Sets

We consider a subset A of the Euclidean space R_m and a set E contained in A. Let $E'_A = A - E$ be the complement of E relative to A, and x a point of A. Then x is called an *internal point* of E (relative to A) whenever $\rho(x, E'_A) > 0$, x is called an *external point* of E (relative to A) whenever it is an internal point of E'_A (that is, whenever $\rho(x, E) > 0$), and x is said to be on the *boundary* (relative to A) of E and also on the boundary of E'_A whenever it is neither an internal nor an external point of E, that is, whenever $\rho(x, E) = \rho(x, E'_A) = 0$. Evidently every set can only contain internal points and points on its boundary.

The set E is called *closed* (relative to A) when it contains its boundary (relative to A). The set E is called *open* (relative to A) when it contains no point of its boundary (relative to A), that is, whenever it contains only internal points (relative to A). The boundary of A itself is considered to be empty, and A is therefore closed as well as open relative to itself.

We may replace the set A in these definitions by the whole space R_m, and it is easily seen that a set E is closed in the sense described earlier (E is closed whenever $E^+ \subset E$) if and only if E is closed relative to R_m. Indeed, let us assume first that $E^+ \subset E$, and let x be on the boundary of E. Then $\rho(x, E) = 0$, which shows that either $x \in E$ or $x \in E^+$, so that on account of $E^+ \subset E$ we find $x \in E$ in any case. The set E contains therefore its boundary. Conversely, if E contains its boundary and

$x \in E^+$, then $\rho(x, E) = 0$, so that x is an internal point of E or on its boundary; hence $x \in E$ in any case. This implies $E^+ \subset E$.

Theorem 1. *The set $E \subset A$ is closed relative to A if and only if its complement $E'_A = A - E$ is open relative to A.*

Proof. Follows immediately from the definitions.

Theorem 2. *Sum and product of a finite number of open sets are open. Sum and product of a finite number of closed sets are closed.*

Proof. 1°. Let $x \in \Sigma O_n$, where all O_n are open. Then $x \in O_n$ for at least one n, and with the point x a whole neighbourhood of x (as far as this neighbourhood is contained in the set A, relative to which O_n is open) is contained in the set O_n. This neighbourhood is therefore also contained in ΣO_n, in other words, x is an internal point of ΣO_n. It follows that ΣO_n is open.

2°. Consider two open sets O_1 and O_2 with complements O'_1 and O'_2 (relative to a set A or to the whole space R_m). If x is a point of O_1O_2, then $\rho(x, O'_1) > 0$ and $\rho(x, O'_2) > 0$, hence $\rho(x, O'_1 + O'_2) > 0$ or, since $O'_1 + O'_2 = (O_1O_2)'$ by § 1, Th. 5, $\rho\{x, (O_1O_2)'\} > 0$. This shows that x is an internal point of O_1O_2, so that O_1O_2 is open. The extension to the product of more than two open sets is evident.

3°. If $F_1, \ldots, F_n$ are closed sets, the complements $F'_1, \ldots, F'_n$ are open, so that, by what we have already proved, $\Pi_{i=1}^n F'_i$ is also open. This implies that $\Sigma_{i=1}^n F_i = (\Pi_{i=1}^n F'_i)'$ is closed.

4°. If once more $F_1, \ldots, F_n$ are closed, then $\Sigma_{i=1}^n F'_i$ is open, so that $\Pi_{i=1}^n F_i = (\Sigma_{i=1}^n F'_i)'$ is closed.

Theorem 3. *The sum of an enumerable infinity of open sets is open, and the product of an enumerable infinity of closed sets is closed.*

Proof. In order to prove the first statement, we have only to repeat the proof of the first part of the preceding theorem.

Let now $F_1, F_2, \ldots$ be closed. Then their complements $F'_1, F'_2, \ldots$ are open, so that $\Sigma_{i=1}^\infty F'_i$ is also open. Hence its complement $\Pi_{i=1}^\infty F_i$ is closed.

Remark. The sum of an enumerable infinity of closed sets is not necessarily closed. Considering for example the sum of all points with rational coordinates in the linear interval [0, 1], we have a sum of closed sets which is not closed itself. It follows that the product of an enumerable infinity of open sets is not necessarily open.

Theorem 4. *If the sets F_1 and F_2 are closed relative to the whole space R_m and have no common points, while one at least of them, say F_1, is bounded, then* $\rho(F_1, F_2) > 0$.
Proof. Let us suppose that $\rho(F_1, F_2) = 0$. Then there exist two sequences of points $x_n \in F_1$ and $y_n \in F_2$ such that $\lim \rho(x_n, y_n) = 0$. The set F_1 being bounded, the sequence x_n contains, by the Bolzano-Weierstrass Theorem, a subsequence z_n $(z_1 = x_{n_1}, z_2 = x_{n_2}, \ldots)$ converging to a point z. Hence, if t_n is the corresponding subsequence of y_n, we have $\lim \rho(z, z_n) = 0$ and $\lim \rho(z_n, t_n) = 0$. Since $\rho(z, t_n) \leqslant \rho(z, z_n) + \rho(z_n, t_n)$, this implies $\lim \rho(z, t_n) = 0$. From $\lim \rho(z, z_n) = 0$, $z_n \in F_1$ it follows now that $z \in F_1$, whereas $\lim \rho(z, t_n) = 0$, $t_n \in F_2$ shows that $z \in F_2$. This is impossible, F_1 and F_2 having no common points. Hence $\rho(F_1, F_2) > 0$.

Remark. If neither F_1 nor F_2 is bounded, the theorem is not necessarily true, e.g. in the two-dimensional case that F_1 is the set of all points (x, y) such that $x < 0$, $y \geqslant -x^{-1}$ and F_2 is the set of all points (x, y) such that $x > 0$, $y \geqslant x^{-1}$.

§ 4. Nets. Decomposition of an Open Set

If Δ is a closed interval in R_m, we call *net of closed intervals* on Δ any sequence of closed non-overlapping intervals whose sum is identical with Δ. By *net of cells* on R_m we mean a sequence of cells no two of which have common points and whose sum covers R_m. A sequence N_k $(k = 1, 2, \ldots)$ of nets is called *regular* if each interval of N_{k+1} is contained in an interval of N_k and if the characteristic number of N_k (cf. § 2) tends to zero as $k \to \infty$.

Theorem 1. *A set E, which is open relative to a closed interval Δ, is the sum of a sequence of closed non-overlapping intervals. A set E, which is open relative to R_m, is the sum of a sequence of cells without common points.*
Proof. Let E be open relative to Δ, and let N_k be a regular sequence of nets of closed intervals on Δ, such that every net N_k contains only a finite number of intervals. Let, furthermore, M_1 be the sum of those intervals of N_1 that lie in E, and let M_k, for $k \geqslant 2$, be the sum of those intervals of N_k that lie in E, but not in any of the preceding $M_l (l < k)$. Since the characteristic number of N_k tends to zero and E is open, the enumerable collection of intervals ΣM_k covers E.

Given a set E, open relative to R_m, we may repeat the proof for cells without common points, using now a regular sequence of nets of cells on R_m.

§ 5. The Heine-Borel-Lebesgue Covering Theorem

It is sometimes convenient to know under what conditions a point set which is covered by an infinity of other point sets, may already be covered by a finite number of these sets. The theorem which follows now gives sufficient conditions for this situation to arise.

Theorem 1 (*Heine-Borel-Lebesgue Covering Theorem*). *Let F be a set, bounded, and closed relative to R_m, and let S be a collection of sets, open relative to R_m and such that every point of F belongs to one at least of them. Then F is covered by a finite number of the open sets of S.*

Proof. Since F is bounded, there exists a closed interval Δ containing F. The set $\Delta - F$ is open relative to Δ, which implies that every point of $\Delta - F$ is an internal point of a set O, open relative to R_m and having no points in common with F (we may take for example $O = O_1 - F$, where O_1 is an open interval containing Δ). Adding this open set O to the collection S, we obtain a collection T of open sets. Evidently every $x \in \Delta$ is a point of at least one of the sets (T). In order to show now that F is covered by a finite number of the sets (S), it is clearly sufficient to prove that Δ is covered by a finite number of the sets (T). For this purpose we consider a regular sequence N_k of nets of closed intervals on Δ, each net N_k containing only a finite number of intervals. The theorem will be proved once we have shown the existence of an index k such that each of the finite number of intervals of N_k is contained in a set (T). If such an index k did not exist, we could take for every value of k an interval Δ_k of N_k, not contained in a set (T). The centres x_k of the intervals Δ_k (whose diameters tend to zero) would have, in virtue of the Bolzano-Weierstrass Theorem, at least one point of accumulation x belonging to Δ, and in every neighbourhood of x there would be consequently an interval Δ_k not covered by a set (T). This however is impossible since x is an internal point of one of the sets (T).

EXAMPLES

1) In § 2 we have used that if x, y and z are three arbitrary points in Euclidean space R_m, then $\rho(x, y) < \rho(x, z) + \rho(y, z)$. Prove this inequality.

(It is no restriction of the generality to suppose that z has all its coordinates equal to zero. From $ab \leqslant (a^2 + b^2)/2$, holding for $a, b \geqslant 0$, we derive, taking $a = |x_i|/(\Sigma x_i^2)^{1/2}$, $b = |y_i|/(\Sigma y_i^2)^{1/2}$ and summing over i, that $\Sigma |x_i y_i| \leqslant (\Sigma x_i^2 . \Sigma y_i^2)^{1/2}$. Hence, replacing y_i by $x_i - y_i$,

$$\rho^2(x, y) = \Sigma (x_i - y_i)^2 \leqslant \Sigma |x_i| . |x_i - y_i| + \Sigma |y_i| . |x_i - y_i| \leqslant \{(\Sigma x_i^2)^{1/2} + (\Sigma y_i^2)^{1/2}\} \{\Sigma (x_i - y_i)^2\}^{1/2},$$

from which the result follows).

2) In connection with the Heine-Borel-Lebesgue Covering Theorem, it may be proved that in each of the following cases the set T is not covered by a finite number of sets of the collection S:

(*a*) T is the open interval $(0, 1)$, S is the collection of all open intervals $(x, 1)$ such that $0 < x < 1$.

(*b*) $T = R_1$ (the whole straight line), S is the collection of all open linear intervals.

(*c*) T is the set consisting of the points $x_0, x_1, x_2, \ldots,$ where $x_0 = 0$, $x_n = 1/n$ $(n = 1, 2, \ldots)$, S is the collection of all sets X_n, where X_n consists of the point x_n only.