Modern University Algebra

MARVIN MARCUS

HENRYK MINC

Modern University Algebra

MARVIN MARCUS
HENRYK MINC
University of California, Santa Barbara

THE MACMILLAN COMPANY · NEW YORK
COLLIER-MACMILLAN LIMITED · LONDON

First Printing

Library of Congress catalog card number: 66-19712

THE MACMILLAN COMPANY, NEW YORK
COLLIER-MACMILLAN CANADA, LTD., TORONTO, ONTARIO

PRINTED IN THE UNITED STATES OF AMERICA

Modern University Algebra

To

Rebecca, Robert, Ralph,

and Raymond

Preface

This book is intended for use in a one-semester or two-quarter course in algebra at the freshman or sophomore level. We assume only that the student has completed the standard high school courses in algebra and trigonometry, although the latter is not essential. At the University of California, Santa Barbara, this material has been used to replace the one-semester course which is ordinarily entitled "Advanced College Algebra" or "Theory of Equations." In this department such a course has traditionally been taught to second semester freshmen or beginning sophomores and is a requirement for all mathematics majors. It is generally agreed that some of the material in a classical course in the theory of equations is no longer pertinent to the educational requirements of mathematics and science majors. In Chapter 4 of this book, however, we have tried to retain those important parts of the theory of equations that bear some relation to other branches of mathematics–for example, linear algebra, numerical analysis, and modern algebra.

The goal of this book is to include those important topics in algebra that are usually assumed to be too elementary to be incorporated in a serious treatment of "modern algebra" (à la van der Waerden or Jacobson) and too advanced to be called "college algebra" in the ordinary sense. It is hoped that this material will prepare the student for subsequent courses in linear algebra, modern algebra, and some parts of analysis–for example, the theory of convex sets and convex functions. Our experience at Santa Barbara has been that many students majoring in the social sciences, as well as in the natural sciences, have found this material to be useful. This is particularly true of Chapters 2 and 3, on combinatorial analysis and convexity.

We have not tried in any sense to develop an axiomatic treatment of the material contained herein; rather, we have tried to penetrate significantly into the actual subject matter of the topics that we have dealt with. It is perhaps best to illustrate our viewpoint by example. In Chapter 2, on combinatorial analysis, we use the fact that a positive integer can divide another positive integer to yield a unique quotient and a nonnegative remainder. This fact is familiar to every fifth-grade student and thus we feel that it can safely be used in Chapter 2, although it is not until Chapter 4 that we give a systematic treatment of elementary number theory, including the Division Algorithm.

Again, we assume in Chapter 1 that the student is acquainted with the most elementary aspects of the "greater than" relation between real numbers although we do not investigate the theory of inequalities until Chapter 3.

It has been our experience that axiomatics at an elementary level is an inappropriate approach to the study of mathematics. There are altogether too many important, new, and exciting ideas in mathematics and the student may not be able to afford the time for a lengthy contemplation of material such as the Peano axioms for the integers. In our opinion, it is much more important to know something about convex sets, convex functions, and the classical inequalities than to be preoccupied with an interminable sequence of trivial results culminating in $(-1) \times (-1) = 1$.

The experienced teacher will see from looking at the table of contents that many of the topics that are covered here seem quite advanced for the ordinary student for whom this text is intended. The Frobenius-König theorem, the Minkowski inequality, and the basis theorem for symmetric polynomials are examples of topics that might well be deferred to a more advanced course. However, we have presented this material at a level which we feel makes it accessible to students with backgrounds such as we have described above. Of course, we also include much of the traditional material, such as de Moivre's theorem, greatest common divisors, and the remainder theorem for polynomials.

The four chapters are essentially self-contained and, in the main, independent of one another. Chapter 1, "Numbers and Sets," deals with the elementary language of mathematics: induction, summation and product notation, composition of functions, elementary theory of cardinality. Chapter 2, "Combinatorial Analysis," gives a fairly complete theory of permutations on a finite set. In this chapter we also introduce the concept of an incidence matrix and we motivate the definitions of matrix product and sum by their combinatorial applications. Although we state and prove the important elementary properties of matrices, we do not attempt to give a systematic treatment of linear algebra and the theory of matrices in this text. Chapter 3, "Convexity," has as its goal the investigation of important and classical inequalities for real numbers; for example, the arithmetic-geometric mean inequality, the triangle inequality, and Minkowski's inequality. It is in this chapter that our treatment is somewhat novel. We do not make the development dependent on calculus, and it should be interesting to the knowledgeable reader to see that most of the important aspects of this subject can be done completely independently of the techniques of analysis. Chapter 4, "Rings," unifies the theory of polynomials, integers, and complex numbers by regarding them as similar algebraic structures. The last section, on the theory of equations, lies on the boundary between analysis and algebra, and because of this, some of the arguments that are used in the development of the Sturm theory have a strong analytical flavor. The single item that is used from analysis, the least-upper-bound axiom, is explicitly stated and the immediate consequence of this axiom that polynomial functions attain their maxima and minima on closed intervals is also explicitly set out.

Every section of the book ends with a true-false quiz and a set of exercises. A complete set of solutions and explanations is provided in the "Answers and Solutions" at the end of the book. We have taken the attitude that any serious student will make a genuine attempt to solve some of the more difficult questions by himself and that, if he is unsuccessful, the immediate accessibility of a solution can only serve to reinforce the learning process. The authors have found in teaching this material that the first three chapters can be covered without difficulty in about three quarters of a semester. This means that in an ordinary one-semester course a certain amount of material in Chapter 4 would have to be omitted. We suggest omitting Section 5 and parts of Section 6; in particular, the Sturm theory. Also, for the better-trained high school graduate, much of the material in Chapter 1 will be repetitious and could therefore be covered rather quickly or omitted altogether. We would suggest that a minimum syllabus for a course at this level should include the first three sections of Chapter 1, Sections 1 and 2 of Chapter 2, all of Chapter 3, and Sections 1, 2, 3 and parts of 4 and 6 in Chapter 4. Such a selection of material can be easily covered in a quarter.

The authors are pleased to express their thanks to Dr. David Outcalt, Mrs. Ruth Afflack, and Miss Elizabeth Rau for their many helpful criticisms.

M. M.
H. M.

Contents

Chapter 1. Numbers and Sets 1

1 Mathematical Induction 1
   Σ and Π notation. Factorial. Binomial coefficient. Binomial theorem. Quiz and exercises.

2 Sets 7
   Subset. Union. Intersection. Cartesian product. Power set. Equivalence relation. Quiz and exercises.

3 Functions 14
   One-one. onto. Inverse function. Composition of functions. Quiz and exercises.

4 Cardinality 20
   Equipotent sets. Denumerable sets. Quiz and exercises.

Chapter 2. Combinatorial Analysis 27

1 Introduction to permutations 27
   Cycles. Quiz and exercises.

2 Parity 35
   Transpositions. Cauchy number. Sign of a permutation. Permuted sequences. Quiz and exercises.

3 Incidence Matrices 41
   Matrix products. Permutation matrix. Transpose. Sum of matrices. Diagonals. Frobenius-König theorem. Quiz and exercises.

4 Birkhoff's Theorem 54
   Doubly stochastic matrices. Quiz and exercises.

Chapter 3. Convexity 63

1 Convex Sets 63
   Space of real n-tuples. Convex set. Convex hull. Convex polyhedron. Linear function. Linear programming. Quiz and exercises.

2 Elementary Inequalities 77
   The Arithmetic—geometric mean inequality. Quadratic inequality. Cauchy's inequality. Triangle inequality. Absolute value. Rearrangement theorem. Quiz and exercises.

3 Convex Functions and Inequalities 91
Convexity of $f(x) = x^n$. Convex cone. Homogeneous function. Minkowski inequality. Convex function on a convex polyhedron. Convex non-decreasing function of a convex function. Quiz and exercises.

Chapter 4. Rings 108

1 Introduction to Rings 108
Rings. Properties of rings. Isomorphism. Units. Quiz and exercises.

2 Integers 119
Primes. Divisibility. Division algorithm. Greatest common divisor. Euclidean algorithm. Fundamental theorem of arithmetic. Congruences. Residue classes. Linear congruence equations. Quiz and exercises.

3 Complex Numbers 132
Real and imaginary parts. Conjugates. Argand diagram. Radian measure. Trigonometric functions. Amplitude. De Moivre's theorem. Roots of a complex number. Rational powers of a complex number. Quiz and exercises.

4 Polynomials 147
Indeterminates. Polynomial notations. Degree. Division algorithm. Remainder theorem. Right- and left-hand values of a polynomial. Divisibility. Greatest common divisors. Euclidean algorithm. Unique factorization theorem. Quiz and exercises.

5 Construction of an Indeterminate 168
The set R^X. Existence of an indeterminate. Quiz and exercises.

6 Theory of Equations 180
Polynomials in several indeterminates. Symmetric polynomials. Elementary symmetric functions. Lexicographic order. Basis theorem for symmetric polynomials. Relation between roots and coefficients. Fundamental theorem of algebra. Derivative of a polynomial. Multiple zeros. Rational roots of a polynomial. Sturm sequence. Sturm functions. Solution of cubic and quartic equations. Quiz and exercises.

ANSWERS AND SOLUTIONS 206

INDEX 241

1

Numbers and Sets

1.1 Mathematical Induction

This book is intended for use in an algebra course at the freshman or sophomore level. Any student who has successfully completed the standard high school courses in algebra has sufficient preparation to proceed with this subject matter. Our raw material will be numbers: *real numbers, rational numbers, integers and natural numbers*. May we recall that a rational number is a real number of the form p/q where p and q are integers, $q \neq 0$. We use the term natural number synonymously with the term positive integer. The natural numbers are also called *counting numbers*, since the sequence 1, 2, 3,... is used in enumerating objects or events. The sequence of natural numbers is infinite; that is to say, given any number in this sequence we can conceive of a natural number which immediately follows it.

Many propositions in mathematics are proved or disproved by considering all possible cases. This often cannot be done if the proposition involves the infinite sequence of natural numbers since the number of cases to be considered may well be infinite. However, we sometimes can prove such propositions by using the *principle of mathematical induction*. This principle is inherent in the concept of natural numbers, and we do not attempt to prove it by means of simpler mathematical notions.

Principle of Mathematical Induction *Let a proposition $P(n)$ be either true or false for every natural number n. Suppose that*

(i) *$P(1)$ is true,*

(ii) *the truth of $P(k)$ implies the truth of $P(k+1)$ for all k.*

Then $P(n)$ is true for all natural numbers n.

Although, as we said, we shall not prove the principle of induction, we can argue the plausibility of this principle. Suppose that both induction hypotheses hold. Then, by (i), $P(1)$ is true. It follows from (ii) (with $k = 1$) that $P(2)$ is true. Another application of (ii) (with $k = 2$) shows that $P(3)$ is true. And so on.

Before we give examples of the use of induction we introduce some notation.

Definition 1.1 (Σ and Π notation) *The symbols Σ and Π are used as an abbreviated notation for addition and multiplication. If $a_1, a_2, \ldots, a_m$ is any finite collection of numbers or other mathematical entities for which addition (multiplication) is defined then*

$$\sum_{i=1}^{m} a_i = a_1 + a_2 + a_3 + \cdots + a_m \tag{1}$$

and

$$\prod_{i=1}^{m} a_i = a_1 a_2 a_3 \cdots a_m. \tag{2}$$

These definitions can also be stated as follows:

$$\sum_{i=1}^{1} a_i = a_1 \qquad \text{and, for } m > 1, \qquad \sum_{i=1}^{m} a_i = \left(\sum_{i=1}^{m-1} a_i\right) + a_m; \tag{3}$$

$$\prod_{i=1}^{1} a_i = a_1 \qquad \text{and, for } m > 1, \qquad \prod_{i=1}^{m} a_i = \left(\prod_{i=1}^{m-1} a_i\right) a_m. \tag{4}$$

The definitions (3) and (4) are rigorous versions of (1) and (2) and, in fact, define the symbols

$$\sum_{i=1}^{m} a_i \quad \text{and} \quad \prod_{i=1}^{m} a_i$$

inductively. Thus, for example,

$$\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4, \qquad \prod_{i=1}^{4} a_i = a_1 a_2 a_3 a_4,$$

$$\sum_{i=1}^{4} (2i + 1) = (2 \times 1 + 1) + (2 \times 2 + 1) + (2 \times 3 + 1) + (2 \times 4 + 1) = 24,$$

$$\prod_{i=1}^{4} (2i + 1) = (2 \times 1 + 1)(2 \times 2 + 1)(2 \times 3 + 1)(2 \times 4 + 1) = 945,$$

$$\sum_{i=1}^{4} b^i = b + b^2 + b^3 + b^4, \qquad \prod_{i=1}^{4} b^i = bb^2b^3b^4 = b^{10},$$

$$\sum_{j=1}^{4} a_i b_j = a_i b_1 + a_i b_2 + a_i b_3 + a_i b_4 = a_i(b_1 + b_2 + b_3 + b_4),$$

$$\prod_{t=1}^{4} a_i^t = a_i a_i^2 a_i^3 a_i^4 = a_i^{10}, \qquad \prod_{i=1}^{4} a_i^t = a_1^t a_2^t a_3^t a_4^t.$$

We are now in a position to give examples of the use of the principle of mathematical induction.

EXAMPLE 1.1

Prove that

$$\sum_{i=1}^{n} (2i - 1) = n^2. \tag{5}$$

We use induction on n. If $n = 1$ then both sides of (5) are equal to 1 and thus (5) is true for $n = 1$. We assume now that (5) is true for $n = k$ and show that this assumption implies that (5) holds for $n = k + 1$. Now, if

$$\sum_{i=1}^{k} (2i - 1) = k^2$$

then

$$\begin{aligned}\sum_{i=1}^{k+1} (2i - 1) &= \sum_{i=1}^{k} (2i - 1) + (2(k + 1) - 1)\\ &= k^2 + 2k + 1\\ &= (k + 1)^2.\end{aligned}$$

Thus we have proved by induction that (5) holds for all n.

Suppose we can prove that a proposition $P(h)$ is true for some fixed h and that the assumption of $P(k)$ being true implies that $P(k + 1)$ is true for any $k \geq h$. Then clearly $P(n)$ is true for all $n \geq h$.

EXAMPLE 1.2

Prove that if a is any positive number then

$$(1 + a)^n > 1 + na \tag{6}$$

for any natural number n, $n \geq 2$. Again, we use induction on n. For $n = 2$ we check directly that

$$(1 + a)^2 = 1 + 2a + a^2 > 1 + 2a\,.$$

Assume that

$$(1 + a)^k > 1 + ka.$$

Then it follows that

$$\begin{aligned}(1 + a)^{k+1} &= (1 + a)^k(1 + a)\\ &> (1 + ka)(1 + a)\\ &= 1 + (k + 1)a + ka^2\\ &> 1 + (k + 1)a.\end{aligned}$$

Thus (6) holds for $n = 2, 3, \ldots$.

As our next application of the principle of induction we prove the binomial theorem. Before we do this we shall introduce the notation for factorials and binomial coefficients.

Definition 1.2 (Factorial) *If n is any nonnegative integer we inductively define the number n!, called n-factorial, by*

$$0! = 1 \qquad \textit{and, for } n > 0, \qquad n! = (n-1)!\, n.$$

Using more suggestive, if somewhat vaguer, notation,

$$n! = 1 \times 2 \times 3 \times \cdots \times n$$

$$= \prod_{i=1}^{n} i, \qquad n > 0.$$

Definition 1.3 (Binomial coefficient) *If n and r are integers, $0 \le r \le n$, we define*

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}.$$

The number $\binom{n}{r}$ is called the binomial coefficient n over r.

The following properties of binomial coefficients follow immediately from the definition:

$$\binom{n}{r} = \binom{n}{n-r}, \tag{7}$$

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}. \tag{8}$$

For example, we can establish (8) by computing

$$\binom{n}{r} + \binom{n}{r+1} = \frac{n!}{(n-r)!\, r!} + \frac{n!}{(n-r-1)!\, (r+1)!}$$

$$= \frac{n!\,(r+1) + n!\,(n-r)}{(n-r)!\,(r+1)!}$$

$$= \frac{n!\,(n+1)}{(n-r)!\,(r+1)!}$$

$$= \frac{(n+1)!}{(n+1-(r+1))!\,(r+1)!}$$

$$= \binom{n+1}{r+1}.$$

Formula (8) suggests a method for constructing a table of binomial coefficients:

n \ r	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1
.	.	.	.	.	.	.

This table is called the *Pascal Triangle*.

Theorem 1.1 (Binomial theorem) *If a and b are real numbers and n is a natural number then*

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r. \tag{9}$$

Proof. We use induction on n. Formula (9) clearly holds for $n = 1$. We assume that (9) holds for $n = k$; i.e., that

$$(a+b)^k = \sum_{r=0}^{k} \binom{k}{r} a^{k-r} b^r.$$

We shall prove that this assumption implies the truth of (9) for $n = k + 1$. We use our assumption to compute

$$\begin{aligned}(a+b)^{k+1} &= (a+b)(a+b)^k \\ &= (a+b)\sum_{r=0}^{k}\binom{k}{r}a^{k-r}b^r \\ &= \sum_{r=0}^{k}\binom{k}{r}a^{k-r+1}b^r + \sum_{r=0}^{k}\binom{k}{r}a^{k-r}b^{r+1}.\end{aligned}$$

Now, set $t = r + 1$ in the second sum. It becomes

$$\sum_{t=1}^{k+1}\binom{k}{t-1}a^{k-t+1}b^t$$

which is, of course, equal to

$$\sum_{r=1}^{k+1}\binom{k}{r-1}a^{k-r+1}b^r.$$

Therefore,

$$\begin{aligned}(a+b)^{k+1} &= \sum_{r=0}^{k}\binom{k}{r}a^{k-r+1}b^r + \sum_{r=1}^{k+1}\binom{k}{r-1}a^{k-r+1}b^r \\ &= \binom{k}{0}a^{k+1} + \sum_{r=1}^{k}\left(\binom{k}{r}+\binom{k}{r-1}\right)a^{k-r+1}b^r + \binom{k}{k}b^{k+1} \\ &= \binom{k+1}{0}a^{k+1} + \sum_{r=1}^{k}\binom{k+1}{r}a^{k+1-r}b^r + \binom{k+1}{k+1}b^{k+1},\end{aligned}$$

where we have used (8) and the fact that $\binom{m}{m} = \binom{m}{0} = 1$ for any natural number m. Thus

$$(a+b)^{k+1} = \sum_{r=0}^{k+1}\binom{k+1}{r}a^{k+1-r}b^r$$

and the proof by induction is complete.

Quiz

Answer **true** or **false**:

(In what follows, r and n are positive integers.)

1. $\sum_{t=1}^{4} t^t = \sum_{t=2}^{5}(t+1)^{t+1}$.

2. $\prod_{k=0}^{5}\cos(k\pi/6) = 0$.

3. $\sum_{k=0}^{5}\cos(k\pi/6) = 0$.

4. If $r < n$, then $\binom{n}{r} = \prod_{t=1}^{r}\frac{n-t+1}{t}$.

5. For any r, $(r^2)! = (r!)^2$.

6. For any r, $\binom{2r}{r} \geq 2^r$.

7. If a and b are nonzero real numbers and $(a+b)^n = a^n + b^n$, then $n = 1$.

8. If $2r+1 < n$ then $\binom{n}{r} < \binom{n}{r+1}$.

9. $\binom{n}{r} < \binom{n+1}{r+1}$ for all $n > r \geq 0$.

10. For real numbers the coefficient of x^4y^2 in the expansion of $(x+y)^6$ is $\binom{4}{2} = 6$.

Exercises

1. Compute the following numbers:

(a) $\sum_{t=2}^{5}(-t)^2$; (b) $\sum_{t=1}^{4}(2t)$; (c) $\sum_{k=2}^{2}\frac{1}{k}$;

(d) $\sum_{n=0}^{3}(-1)^n\frac{2^{2n}}{(2n)!}$; (e) $\sum_{s=1}^{20}s^2/\sum_{t=1}^{20}t^2$;

(f) $\sum_{s=1}^{4}s!/\sum_{t=2}^{4}t!$; (g) $(\prod_{s=1}^{10}s!)/(\prod_{t=3}^{10}t!)$;

(h) $\prod_{i=1}^{4}(2i-1)$; (i) $\prod_{i=1}^{15}\frac{2i-1}{2i+1}$.

2. Write out the following expressions in full:

(a) $\sum_{n=1}^{4}(-1)^n\frac{x^{2n-1}}{(2n-1)!}$; (b) $(1-x)^6$;

(c) $(1-x)\sum_{t=0}^{5}x^t$ $(x^0=1)$.

3. Find the value of $\binom{21}{19}$.

4. If $\binom{n}{3}=\frac{10}{21}\binom{n}{5}$, find n.

5. Find the coefficient of x^{17} in the expansion of $(x-2)^{20}$.

6. Prove that $\sum_{r=0}^{n}\binom{n}{r}=2^n$ and that $\sum_{r=0}^{n}(-1)^r\binom{n}{r}=0$.

7. Expand the expression $(x-1/x)^7$ in decreasing powers of x.

8. Show that $n^n\geq(n+1)!$ for $n\geq 3$.

9. Prove that $\binom{n}{r}=\binom{n-2}{r}+2\binom{n-2}{r-1}+\binom{n-2}{r-2}$.

10. For a fixed n find the greatest value of $\binom{n}{r}$.

11. Prove that $(n/2)^n > n!$ for $n\geq 6$.

1.2 Sets

The concept of *set* is a ubiquitous one in mathematics. It is, however, usually left undefined. By a set of mathematical objects, which we call the *elements* of the set, we understand the *totality* (or the *collection*, or the *aggregate*, or the *class*) of these elements. For example, we can speak about the set of natural numbers, the set of real numbers, the set of integers greater than 4, the set of points inside a given circle, the set of solutions of an equation,