

Foundations of
Boundary Layer
Theory for
Momentum,
Heat, and
Mass Transfer

JOSEPH
—A.—
SCHETZ

Joseph A. Schetz

*Aerospace and Ocean Engineering Department
Virginia Polytechnic Institute and State University*

FOUNDATIONS OF BOUNDARY LAYER THEORY

for Momentum, Heat, and Mass Transfer

PRENTICE-HALL, INC.
Englewood Cliffs, New Jersey 07632

Library of Congress Cataloging in Publication Data

SCHETZ, JOSEPH A.

Foundations of boundary layer theory for momentum, heat, and mass transfer.

Bibliography: p.
Includes index.

1. Boundary layer. 2. Momentum transfer. 3. Heat—Transmission. 4. Mass transfer. I. Title.

QA913.S35 1984 532'.05 83-7318

ISBN 0-13-329334-3

© 1984 PRENTICE-HALL, INC.
Englewood Cliffs, N.J. 07632

All rights reserved. No part of this book
may be reproduced in any form or by any means
without permission in writing from the publisher.

Printed in the United States of America

10 9 8 7 6 5 4 3

ISBN: 0-13-329334-3

Editorial/production supervision
and interior design by Tom Aloisi
Cover design by Ray Lundgren
Manufacturing buyer: Tony Caruso

Prentice-Hall International, Inc., *London*
Prentice-Hall of Australia Pty. Limited, *Sydney*
Prentice-Hall Canada Inc., *Toronto*
Editora Prentice-Hall do Brazil, Ltda., *Rio de Janeiro*
Prentice-Hall of India Private Limited, *New Delhi*
Prentice-Hall of Japan, Inc., *Tokyo*
Prentice-Hall of Southeast Asia Pte. Ltd., *Singapore*
Whitehall Books Limited, *Wellington, New Zealand*

This book is dedicated to the memory of William H. Webb, builder of clipper ships and founder and endower of the Webb Institute of Naval Architecture, whose generosity made it possible for this grandson of uneducated Albanian, German, and Slovak immigrants to obtain an excellent technical education completely free.

PREFACE

This volume was planned and written as a textbook for advanced undergraduate or beginning graduate students with three primary goals. First, it was intended to present understandable coverage at that academic level of the advances in turbulence modeling and the application of large digital computers to boundary layer problems, which have revolutionized the field over the last two decades. None of the existing texts serve that purpose. Some have held that modern numerical methods cannot be taught to engineering undergraduates so that they finish the course with any usable *tools*. That view is rejected here. Indeed, this writer strongly believes that the modern university student grew up in the *computer age*, and that he or she finds this type of material easier to grasp than such classical topics as Laplace transforms or Fourier series. Repeated experience in the classroom has proven this view to be correct. The second goal was to treat mass transfer in an integrated manner with momentum and heat transfer. The phenomena and methods of analysis are so similar that it seems inefficient and confusing to split convective mass transfer off as a separate subject as has often been the practice. Finally, a determined effort has been made throughout to relate viscous phenomena in general to the real world.

This book is written to be applicable for courses for mechanical, aerospace, chemical, civil, and ocean engineering students. The treatment presumes that the student has had at least one undergraduate course in fluid mechanics. Tables following this preface suggest coverage for a one-semester or two-quarter course for different majors at both the advanced undergraduate and beginning graduate levels.

To achieve the goals set for coverage and length, it was necessary to omit any discussions of unsteady flows or truly three-dimensional cases. Those topics are, however, usually not discussed in any detail in courses at the intended level. Further, to leave room in the book and time in the classroom for thorough treatments of numerical methods and turbulent flows, much of the older material on laminar flows was also omitted. Although some of that material is quite elegant and interesting, it really has little actual use to the practicing engineer.

There is one other somewhat unusual feature to the organization of the material. Integral methods are introduced very early, before the derivation of the differential equations of motion. The purpose here was to provide the student with some tools, so that simple problems can be worked early in the course. The author has found this to be a helpful motivating factor for the student.

A book is a personal thing to any author and it obviously reflects his or her individual background, experience, and current view of the subject. This writer has been fortunate to have had the opportunity to interact with some of the most prolific workers in the field: Robert M. Drake, Jr., George Mellor, Antonio Ferri, Paul Libby, and Edward R. Van Driest. To them, sincere thanks are due. Special thanks are due Roger Eichhorn, who taught me the value of a combined experimental and analytical approach to any new boundary layer problem. Finally, several people were kind enough to read early versions of the manuscript and provide constructive comments. My thanks to David Rooney, Heehwan Lee, George Wills, Dinshaw Contractor, Herman Krier, Felix Pierce, and George Inger.

Joseph A. Schetz
Blacksburg, Va.
July 1982

Upper Division Mechanical or Chemical Engineering	Upper Division Aerospace Engineering	Upper Division Civil or Ocean Engineering
Chap. 1	Chap. 1	Chap. 1
Chap. 2	Chap. 2	Chap. 2
Chap. 3	Chap. 3	(Skip Secs. 2-4, 2-5, 2-6, and 2-7)
Chap. 4	Chap. 4	Chap. 3
Skip Chap. 5	Chap. 5	(Skip Secs. 3-4 and 3-5)
Chap. 6	(Skip Secs. 5-4, 5-6, and 5-8-1)	Chap. 4
(Skip Secs. 6-2, 6-3, and 6-5-6)	Chap. 6	(Skip Secs. 4-4 and 4-5)
Chap. 7	(Skip Secs. 6-2 and 6-3)	Skip Chap. 5
(Skip Secs. 7-10, 7-11, 7-12, 7-13, and 7-14)	Chap. 7	Chap. 6
Chap. 8	(Skip Secs. 7-10, 7-11, 7-12, 7-13, and 7-14)	(Skip Secs. 6-2, 6-3, and 6-5-6)
(Skip Secs. 8-4, 8-5, 8-6, and 8-7)	Skip Chap. 8	Chap. 7
Chap. 9	Chap. 9	(Skip material on injection and Secs. 7-10, 7-11, 7-12, 7-13, and 7-14)
(Skip high-speed-flow material in Secs. 9-2-1 and 9-2-2, and Secs. 9-3-3, 9-3-4, and 9-3-5)	(Skip Secs. 9-3-3, 9-3-4, and 9-3-5)	Chap. 8
		(Skip Secs. 8-2-2, 8-2-3, variable-density material, 8-3-2, 8-4, 8-5, 8-6, and 8-7)
		Skip Chap. 9
		<i>Ocean Engineering</i>
		Add outside material on free surface effects
		<i>Civil Engineering</i>
		Add outside material on open channel flows and sedimentation

First-Year Graduate Mechanical or Chemical Engineering	First-Year Graduate Aerospace Engineering	First-Year Graduate Civil or Ocean Engineering
Chap. 1 (Review only)	Chap. 1 (Review only)	Chap. 1 (Review only)
Appendix B	Appendix B	Appendix B
Skip Chap. 2 (<i>or</i> Review only)	Skip Chap. 2 (<i>or</i> Review only)	Skip Chap. 2 (<i>or</i> Review only)
Chap. 3 (Review only)	Chap. 3 (Review only)	Chap. 3 (Review only)
Chap. 4	Chap. 4	(Skip Secs. 3-4 and 3-5)
Skip Chap. 5	Chap. 5	Chap. 4
Chap. 6	Chap. 6	Skip Chap. 5
(Skip Sec. 6-5-6)	Chap. 7	Chap. 6
Chap. 7	(Review only Secs. 7-3-1, 7-3-2, and 7-7)	(Skip Sec. 6-5-6)
(Review only Secs. 7-3-1, 7-3-2, and 7-7)	Chap. 8	Chap. 7
Chap. 8	Chap. 9	(Skip material on injection, review only Secs. 7-3-1, 7-3-2, 7-7)
Chap. 9		Chap. 8
(Skip high-speed-flow material)		(Skip Secs. 8-2-2 and 8-2-3 and variable-density material in Sec. 8-3-2)
		Skip Chap. 9
		<i>Civil Engineering</i>
		<i>Add</i> advanced outside material on open channel flow and sedimentation
		<i>Ocean Engineering</i>
		<i>Add</i> advanced outside material on free surface effects

NOTATION

a	Speed of sound and amplification factor
A	Area or constant
$b_{1/2}$	Half-width
B_1, B_2, B'_1, B'_2	Constants
c	Average speed of molecules
c'_i	Fluctuating value of species concentration
c_i	Species concentration
C_i	Mean value of species concentration
$C_1, C_2, \text{etc.}$	Constants
c_p	Specific heat at constant pressure
c_v	Specific heat at constant volume
c_r, c_i	Real and imaginary parts of the phase velocity
C_f	Skin friction coefficient
\bar{C}_f	Average skin friction coefficient
C_p	Pressure coefficient
C_D	Drag coefficient
D, d	Diameter
D_{ij}	Binary diffusion coefficient
D_T	Turbulent diffusion coefficient
e	Internal energy
$f(\cdot)$	Function of (\cdot)

f_i	Body force vector
g	Acceleration of gravity
h	Enthalpy
h	Film coefficient
h_D	Film coefficient for diffusion
$H(\Lambda)$	Shape factor
i	$\equiv \sqrt{-1}$
j	Index
J	Integrated momentum flux
k	Thermal conductivity and average roughness size
k_T	Turbulent thermal conductivity
k_1	Wave number of fluctuations
K_c, K_{cp}	Mass transfer parameters
$K_1, K_2, \text{etc.}$	Constants
K	Turbulent kinetic energy
$E_1(k_1)$	Kinetic energy of axial fluctuations at wave number k_1
ℓ	Turbulent scale length
ℓ_m	Mixing length
Le	Lewis number
Le_T	Turbulent Lewis number
m	Index along surface
\dot{m}_i	Diffusive mass flux of species i
M	Maximum value of m and Mach number
n	Index across layer
N	Maximum value of n
Nu	Nusselt number
Nu_{diff}	Nusselt number for diffusion
p	Pressure
p_i	Partial pressure of species i
P	Mean pressure
P_C, P_T, P_V	Power law decay exponents
p'	Fluctuating pressure
Pr	Prandtl number
Pr_T	Turbulent Prandtl number
\mathcal{P}	Production of turbulent kinetic energy
q_i	Heat flux vector
q_T	Turbulent heat flux
q_w	Wall heat transfer rate

r	Radial coordinate and recovery factor
R	Pipe radius, gas constant, and radius of curvature
Ri	Richardson number
$r_0(x)$	Body radius
$r_{1/2}$	Half-radius
Re	Reynolds number
s	Transformed streamwise coordinate
Sc	Schmidt number
Sc_T	Turbulent Schmidt number
St	Stanton number
St_{diff}	Stanton number for diffusion
$S(\Lambda)$	Shear parameter
t	Time
T	Static temperature
$\mathbf{T}_{x, y, z}$	Surface force vector
T_b	Bulk temperature
T^*	Reference temperature
T_t	Total (stagnation) temperature
T_0	Time period
\bar{T}	Mean temperature
T'	Fluctuating temperature
T_*	Heat transfer temperature
T^+	$\equiv (T_w - \bar{T})/T_*$
u	Streamwise velocity
u_{ave}	Average velocity
U	Mean velocity
\tilde{U}	Mass-weighted mean velocity
u'	Fluctuating velocity
u_*	Friction velocity
u^+	$\equiv U/u_*$
v	Transverse or radial velocity
v_w	Transverse velocity at the wall
V	Mean transverse velocity and general velocity
v_0^+	Dimensionless transverse velocity at the wall
v'	Fluctuating transverse velocity
x	Streamwise coordinate
X_i	Mole fraction of species i
y	Transverse coordinate

Y	Transformed transverse coordinate
$y^+ \equiv yu_*/\nu$	Transverse coordinate for the law of the wall
W_i	Molecular weight
$W(y/\delta)$	Wake function
Z	$\equiv k^m l^n$

Greek

α	Wave number and amplification factor
α_T	$\equiv k_T/\rho c_p$
β	Pressure gradient parameter and wave number
ψ	Planar stream function
Ψ	Axisymmetric stream function
$\hat{\psi}$	Disturbance stream function
ε	Dissipation of turbulent energy
$\varepsilon_{n, m}$	Truncation error
ε_{xy}	Strain
ρ	Density
λ	Pohlhausen pressure gradient parameter, pipe resistance coefficient, and second viscosity coefficient
λ^*	Mean free path between molecules
Λ	Thwaites–Walz pressure gradient parameter
Λ_t	Smith and Spalding parameter
τ, τ_{xy}	Shear
τ_T	Turbulent shear
Ω	Intermittency
μ	Laminar viscosity
μ_T	Turbulent viscosity
κ	Constant in the law of the wall
κ_T	Constant in the Temperature law of the wall
ν	Laminar kinematic viscosity
ν_T	Turbulent kinematic viscosity
ϕ	Amplitude function
δ	Boundary layer thickness
δ_t	Conduction thickness
δ_T	Thermal boundary layer thickness
δ_c	Concentration boundary layer thickness
δ^*, Δ^*	Displacement thickness
δ_k^*	Kinematic displacement thickness

θ	Momentum thickness
Θ, θ_r	Excess temperature
Δ	Clauser integral boundary layer thickness
ζ	$\equiv \delta_T/\delta$
$\phi_{1, 2}$	Deformation angle
Π	Wake parameter
ξ	Dummy variable
η	Similarity variable
$\bar{\eta}$	Transformed transverse coordinate
ω	Dimensionless frequency and viscosity law exponent
$\sigma_{K, z, \tau}$	“Prandtl” numbers for K, Z, τ
Γ	See Eq. (9-8)
γ	Ratio of specific heats

Subscripts

c	Values on the centerline
e	Values at the edge of the boundary layer
j	Initial values in a jet
t	Stagnation values
w	Wall values
∞	Conditions in the approach flow

CONTENTS

PREFACE xiii

NOTATION xvii

1 INTRODUCTION TO VISCOUS FLOWS 1

1-1 The Importance of Viscous Phenomena 1

1-2 Conditions at a Fluid/Solid Boundary 2

1-3 Laminar Transport Processes 3

1-4 The Boundary Layer Concept 9

1-5 Separation and the Kutta Condition 11

1-6 Basic Notions of Turbulent Flow 15

Problems 19

2 INTEGRAL EQUATIONS AND SOLUTIONS FOR LAMINAR FLOW 20

2-1 Introduction 20

2-2 The Integral Momentum Equation 21

2-3 Solution of the Integral Momentum Equation 25

2-3-1 The Pohlhausen Method 25

2-3-2 The Thwaites–Walz Method 28

2-3-3 Flows with Suction or Injection 30

- 2-4 The Integral Energy Equation 31
- 2-5 Solution of the Integral Energy Equation 33
 - 2-5-1 Unheated Starting-Length Problem 33
 - 2-5-2 Nonuniform Wall Temperature 36
 - 2-5-3 Method of Smith and Spalding 37
- 2-6 The Integral Species Conservation Equation 38
- 2-7 Solution of the Integral Species Conservation Equation 39
- 2-8 Relationship of Wall Friction, Heat Transfer, and Mass Transfer 41
- 2-9 Discussion 42
 - Problems 42

3 DIFFERENTIAL EQUATIONS OF MOTION FOR LAMINAR FLOW 44

- 3-1 Introduction 44
- 3-2 Conservation of Mass: The Continuity Equation 45
- 3-3 Conservation of Momentum: The Momentum Equation 47
 - 3-3-1 Modeling of the Laminar Shear Stress 49
 - 3-3-2 Forms of the Momentum Equation for Laminar Flow 50
- 3-4 Conservation of Energy: The Energy Equation 51
 - 3-4-1 Modeling of the Laminar Heat Flux 55
- 3-5 Conservation of Mass of Species: The Species Continuity Equation 56
 - 3-5-1 Modeling of Laminar Diffusion 57
 - 3-5-2 Energy Transfer by Mass Transfer 57
 - 3-5-3 Physical Properties of Mixtures 58
- 3-6 Mathematical Overview 58
 - Problems 61

4 EXACT AND NUMERICAL SOLUTIONS FOR LAMINAR CONSTANT-PROPERTY INCOMPRESSIBLE FLOWS 62

- 4-1 Introduction 62
- 4-2 Fully Developed Flow in a Tube 63
- 4-3 Similar Solutions 64
 - 4-3-1 Exact Solution for Flow over a Flat Plate 66
 - 4-3-2 Similar Solutions with Pressure Gradient 70
- 4-4 Solutions to the Low-Speed Energy Equation 71
 - 4-4-1 Fully Developed Flow in a Tube 71
 - 4-4-2 Similar Solutions 72

- 4-5 Foreign Fluid Injection 75
 - 4-5-1 Similar Solutions 75
 - 4-5-2 Fully Developed Flow in a Tube 78
- 4-6 Numerically Exact Solutions 78
 - 4-6-1 Numerical Analysis of the Linear, Model Equation 82
 - 4-6-2 An Explicit Method for the Boundary Layer Equations 87
 - 4-6-3 An Implicit Method for the Boundary Layer Equations 90
 - 4-6-4 Transformations and Other Matters 90
- Problems 92

5 COMPRESSIBLE LAMINAR BOUNDARY LAYERS 94

- 5-1 Introduction 94
- 5-2 The Adiabatic Wall Temperature 95
- 5-3 The Reference Temperature Method 95
- 5-4 The Special Case of Prandtl Number Unity 96
- 5-5 The Recovery Factor for Nonunity Prandtl Number 99
- 5-6 Compressibility Transformations 101
- 5-7 Exact Solutions for Compressible Flow over a Flat Plate 103
- 5-8 Flows with Mass Transfer 106
 - 5-8-1 The Special Case of $Pr = Le = 1$ 106
 - 5-8-2 Foreign Gas Injection 112
- 5-9 Numerical Solutions of Compressible, Laminar Boundary Layer Flows 112
- 5-10 Pressure Gradients and Separation in High-Speed Flows 115
- Problems 118

6 TRANSITION TO TURBULENT FLOW 119

- 6-1 Introduction 119
- 6-2 Hydrodynamic Stability Theory 121
- 6-3 The e^{10} Method 125
- 6-4 A Method Based on Re_θ 129
- 6-5 Selected Empirical Information 129
 - 6-5-1 The Nature of Transition 129
 - 6-5-2 Free-Stream Turbulence 131
 - 6-5-3 Roughness 131
 - 6-5-4 Bluff Bodies at Low Speeds 132

6-5-5 Density-Stratified Flows 135

6-5-6 Supersonic Flows 136

Problems 138

7 WALL-BOUNDED, INCOMPRESSIBLE TURBULENT FLOWS 139

7-1 Introduction and Scope 139

7-2 Engineering Requirements of Turbulent Analyses 139

7-3 Empirical Information on the Mean Flow as a Basis for Analysis 141

7-3-1 Flow over a Flat Plate 141

7-3-2 Flow in a Pipe 155

7-3-3 Flows with Axial Pressure Gradients 159

7-4 Selected Empirical Turbulence Information 162

7-5 The Central Problem of the Analysis of Turbulent Flows 170

7-6 Mean Flow Turbulent Transport Formulations 174

7-7 Mean Flow Integral Methods 177

7-8 Mean Flow Models for the Eddy Viscosity and the Mixing Length 181

7-8-1 Models for the Inner Region 181

7-8-2 Models for the Outer Region 189

7-8-3 Composite Model for the Whole Boundary Layer 195

7-9 Numerical Solution Methods for Mean Flow Formulations 196

7-10 Formulations Based on Turbulent Kinetic Energy 204

7-11 Formulations Based on Turbulent Kinetic Energy and a Length Scale 212

7-12 Formulations Based Directly on the Reynolds Stress 214

7-13 Direct Turbulence Formulations 217

7-14 Boundary and Initial Conditions for Higher-Order Models 218

Problems 218

8 FREE TURBULENT SHEAR FLOWS—JETS AND WAKES 220

8-1 Introduction 220

8-2 Mean Flow and Turbulence Data for a Round Jet in a Moving Stream 222

8-2-1 Constant-Density Flows 222

8-2-2 Density Variations from Temperature Variations 226

8-2-3 Density Variations from Composition Variations 230