FOURIER SERIES AND BOUNDARY VALUE PROBLEMS

Fourth Edition

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Joseph Fourier

ABOUT THE AUTHORS

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JAMES WARD BROWN is Professor of Mathematics at The University of Michigan—Dearborn. He earned his A.B. in Physics from Harvard University and his A.M. and Ph.D. in Mathematics from The University of Michigan in Ann Arbor, where he was an Institute of Science and Technology Predoctoral Fellow. He is coauthor with Dr. Churchill of the fourth edition of *Complex Variables and Applications*. A recipient of a research grant from the National Science Foundation, he recently received a Distinguished Faculty Award from the Michigan Association of Governing Boards of Colleges and Universities. He is listed in *Who's Who in America*.

JOSEPH FOURIER

JEAN BAPTISTE JOSEPH FOURIER was born in Auxerre, about 100 miles south of Paris, on March 21, 1768. His fame is based on his mathematical theory of heat conduction, a theory involving expansions of arbitrary functions in certain types of trigonometric series. Although such expansions had been investigated earlier, they bear his name because of his major contributions. Fourier series are now fundamental tools in science, and this book is an introduction to their theory and applications.

Fourier's life was varied and difficult at times. Orphaned by the age of 9, he became interested in mathematics at a military school run by the Benedictines in Auxerre. He was an active supporter of the Revolution and narrowly escaped imprisonment and execution on more than one occasion. After the Revolution, Fourier accompanied Napoleon to Egypt in order to set up an educational institution in the newly conquered territory. Shortly after the French withdrew in 1801, Napoleon appointed Fourier prefect of a department in southern France with headquarters in Grenoble.

It was in Grenoble that Fourier did his most important scientific work. Since his professional life was almost equally divided between politics and science and since it was so intimately geared to the Revolution and Napoleon, his advancement of the frontiers of mathematical science is quite remarkable.

The final years of Fourier's life were spent in Paris, where he was Secretary of the Académie des Sciences and succeeded Laplace as President of the Council of the Ecole Polytechnique. He died at the age of 62 on May 16, 1830.

PREFACE

This is an introductory treatment of Fourier series and their applications to boundary value problems in partial differential equations of engineering and physics. It is designed for students who have completed a first course in ordinary differential equations and the equivalent of a term of advanced calculus. In order that the book be accessible to as great a variety of students as possible, there are footnotes referring to texts which give proofs of the more delicate results in advanced calculus that are occasionally needed. The physical applications, explained in some detail, are kept on a fairly elementary level.

The *first objective* of the book is to introduce the concept of orthogonal sets of functions and representations of arbitrary functions in series of functions from such sets. Representations of functions by Fourier series, involving sine and cosine functions, are given special attention. Fourier integral representations and expansions in series of Bessel functions and Legendre polynomials are also treated.

The *second objective* is a clear presentation of the classical method of separation of variables used in solving boundary value problems with the aid of those representations. Some attention is given to the verification of solutions and to uniqueness of solutions; for the method cannot be presented properly without such considerations. Other methods are treated in the authors' book *Complex Variables and Applications* and in the first author's book *Operational Mathematics*.

This book is a revision of the 1978 edition; the first two editions were published in 1963 and 1941 and were written by the first author alone. Considerable attention has been given here to improving the exposition, and there are almost twice as many figures as in the last edition. Also, examples are now clearly labeled as such. There has been some reordering of chapters so that, in this edition, the chapter on boundary value problems involving Fourier series is reached earlier. The theory of Sturm-Liouville problems is now developed after the simpler Sturm-Liouville problems leading to Fourier series have become thoroughly familiar to the reader.

The chapters on Bessel functions and Legendre polynomials, Chapters 8 and 9, are essentially independent of each other and can be taken up in either order.

The last three sections of Chapter 3, on further properties of Fourier series, and Chapter 10, on uniqueness of solutions, can be omitted to shorten the course; this also applies to some sections of other chapters.

In preparing this edition, the authors have benefited from the comments of a variety of people, many of whom are colleagues and students at The University of Michigan. Thanks are also due to R. P. Boas, Jr. and G. H. Brown for furnishing some of the references that are cited in the footnotes; and the derivation of the laplacian in spherical coordinates that is given was suggested by a note of R. P. Agnew's in the *American Mathematical Monthly*, vol. 60 (1953). The authors are especially indebted to V. C. Williams and R. E. Lynch, whose careful reading of the manuscript of this edition led to many improvements, and to P. R. Devine and R. A. Weinstein, who served as editors of the project.

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James Ward Brown

CONTENTS

CHAPTER

ONE

PARTIAL DIFFERENTIAL EQUATIONS OF PHYSICS

1. TWO RELATED TOPICS

This book is concerned with two general topics:

(*a*) one is the representation of an arbitrary function by an infinite series of functions from a prescribed set;

(*b*) the other is a method of solving boundary value problems in partial differential equations, with emphasis on equations that are prominent in physics and engineering.

Representations by series are encountered in solving boundary value problems. The theories of those representations can be presented independently. They have such attractive features as the extension of concepts of geometry, vector analysis, and algebra into the field of mathematical analysis. Their mathematical precision is also pleasing. But they gain in unity and interest when presented in connection with boundary value problems.

The set of functions that make up the terms in the series representation is determined by the boundary value problem. Representations by Fourier series, which are certain types of series of sine and cosine functions, are associated with a large and important class of boundary value problems. We shall give special attention to the theory and application of Fourier series. But we shall also consider extensions and generalizations of such series, concentrating on Fourier integrals and series of Bessel functions and Legendre polynomials.

A boundary value problem is correctly set if it has one and only one solution within a given class of functions. Physical interpretations often suggest boundary conditions under which a problem may be correctly set. In fact, it is sometimes helpful to interpret a problem physically in order to judge whether the boundary conditions may be adequate. This is a prominent reason for associating such problems with their physical applications, aside from the opportunity to illustrate connections between mathematical analysis and the physical sciences.

The theory of partial differential equations gives results on the existence and uniqueness of solutions of boundary value problems. But such results are necessarily limited and complicated by the great variety of types of differential equations and domains on which they are defined, as well as types of boundary conditions. Instead of appealing to general theory in treating a specific problem, our approach will be to actually find a solution, which can often be shown to be the only one possible.

2. LINEAR BOUNDARY VALUE PROBLEMS

In the theory and application of ordinary or partial differential equations, the dependent variable, denoted here by u, is usually required to satisfy some conditions on the boundary of the domain on which the differential equation is defined. The equations that represent those boundary conditions may involve values of derivatives of u, as well as u itself, at points on the boundary. In addition, some conditions on the continuity of u and its derivatives within the domain and on the boundary are required.

Such a set of requirements constitutes a *boundary value problem* in the function u. We apply that term whenever the differential equation is accompanied by some boundary conditions, even though the conditions may not be adequate to ensure a unique solution of the problem.

Example 1. The three equations

$$u''(x) - u(x) = -1 \qquad (0 < x < 1),$$

(1)

$$u'(0) = 0, \quad u(1) = 0$$

constitute a boundary value problem in ordinary differential equations. The differential equation is defined on the domain $0 < x < 1$, whose boundary points are $x = 0$ and $x = 1$. A solution of this problem which, together with each of its derivatives, is continuous on the closed interval $0 \leqslant x \leqslant 1$ is

(2) $$u(x) = 1 - \frac{\cosh x}{\cosh 1}.$$

Solution (2) is easily verified by direct substitution.

Frequently, it is convenient to indicate partial differentiation by writing independent variables as subscripts. If, for instance, u is a function of x and y, we may write

$$u_x \text{ or } u_x(x,y) \text{ for } \frac{\partial u}{\partial x}, \quad u_{xx} \text{ for } \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} \text{ for } \frac{\partial^2 u}{\partial y\, \partial x},$$

etc. We shall always assume that the partial derivatives of u satisfy conditions allowing us to write $u_{yx} = u_{xy}$.

Also, we shall be free to use the symbols $u_x(c,y)$ and $u_{xx}(c,y)$ to denote values of the functions $\partial u/\partial x$ and $\partial^2 u/\partial x^2$, respectively, on the line $x = c$. Corresponding symbols will be used for boundary values of other derivatives.

Example 2. The problem consisting of the partial differential equation

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \qquad (x > 0, y > 0) \tag{3}$$

and the two boundary conditions

$$\begin{aligned} u(0,y) &= u_x(0,y) && (y > 0), \\ u(x,0) &= \sin x + \cos x && (x \geqslant 0) \end{aligned} \tag{4}$$

is a boundary value problem in partial differential equations. The differential equation is defined in the first quadrant of the xy plane. As the reader can readily verify, the function

$$u(x,y) = e^{-y}(\sin x + \cos x) \tag{5}$$

is a solution of this problem. The function (5) and its partial derivatives of the first and second order are continuous in the region $x \geqslant 0, y \geqslant 0$.

A differential equation in a function u, or a boundary condition on u, is *linear* if it is an equation of the first degree in u and derivatives of u. Thus the terms of the equation are either prescribed functions of the independent variables alone, including constants, or such functions multiplied by u or a derivative of u. Note that the general linear partial differential equation of the second order in $u(x, y)$ has the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{6}$$

where the letters A through G denote either constants or functions of the independent variables x and y only.

The differential equations and boundary conditions in Examples 1 and 2 are, evidently, all linear. The differential equation

$$zu_{xx} + xy^2u_{yy} - e^xu_z = f(y,z) \tag{7}$$

is linear in $u(x,y,z)$; but the equation $u_{xx} + uu_y = x$ is nonlinear in $u(x,y)$ because the term uu_y is not of the first degree as an algebraic expression in the two variables u and u_y [compare equation (6)].

A boundary value problem is *linear* if its differential equation and all of its boundary conditions are linear. The boundary value problems in Examples 1 and 2 are, therefore, linear.

The method of solution presented in this book does not apply to nonlinear problems.

A linear differential equation or boundary condition in u is *homogeneous* if each of its terms, other than zero itself, is of the first degree in the function u and its derivatives. Homogeneity will play a central role in our treatment of linear boundary value problems.

Observe that equation (3) and the first of conditions (4) are homogeneous but that the second of those conditions is not. Equation (6) is homogeneous in a domain of the xy plane only when the function G is identically zero ($G \equiv 0$) throughout that domain; and equation (7) is nonhomogeneous unless $f(y,z) \equiv 0$ for all values of y and z being considered.

3. THE VIBRATING STRING

A tightly stretched string, whose position of equilibrium is some interval on the x axis, is vibrating in the xy plane. Each point of the string, with coordinates $(x,0)$ in the equilibrium position, has a transverse displacement $y = y(x,t)$ at time t. We assume that the displacements y are small relative to the length of the string, that slopes are small, and that other conditions are such that the movement of each point is parallel to the y axis. Then, at time t, a point on the string has coordinates $(x,y) = (x,y(x,t))$.

Let the tension of the string be great enough that the string behaves as if it were perfectly flexible. That is, at a point (x,y) on the string, the part of the string to the left of that point exerts a force **T**, in the tangential direction, on the part to the right; and any resistance to bending at the point is to be neglected. The magnitude of the x component of the tensile force **T** is denoted by H. See Fig. 1, where that x component has the same positive sense as the x axis. Our final assumption here is that H is constant. That is, the variation of H with respect to x and t can be neglected.

These idealizing assumptions are severe, but they are justified in many applications. They are adequately satisfied, for instance, by strings of musical instruments under ordinary conditions of operation. Mathematically, the assumptions will lead us to a partial differential equation in $y(x,t)$ that is linear.

Now let $V(x,t)$ denote the y component of the tensile force **T** exerted by the left-hand portion of the string on the right-hand portion at the point (x,y). We take the positive sense of V as that of the y axis. If α is the angle of inclination of

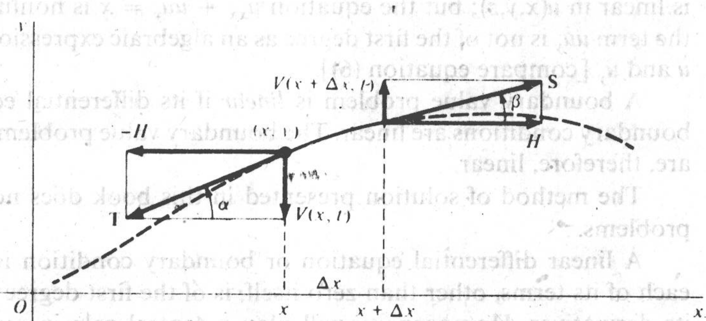

Figure 1

the string at the point (x,y) at time t, then

$$\frac{-V(x,t)}{H} = \tan\alpha = y_x(x,t). \tag{1}$$

This is indicated in Fig. 1, where $V(x,t) < 0$ and $y_x(x,t) > 0$. If $V(x,t) > 0$, then $\pi/2 < \alpha < \pi$ and $y_x(x,t) < 0$; and a similar sketch shows that

$$\frac{V(x,t)}{H} = \tan(\pi - \alpha) = -\tan\alpha = -y_x(x,t).$$

Hence relations (1) still hold. Note, too, that $y_x(x,t) = 0$ when $V(x,t) = 0$, since $\alpha = 0$ then. It follows from relations (1) that *the y component $V(x,t)$ of the force exerted at time t by the part of the string to the left of a point (x,y) on the part to the right is given by the equation*

$$V(x,t) = -Hy_x(x,t) \qquad (H > 0), \tag{2}$$

which is basic for deriving the equation of motion of the string. Equation (2) is also used in setting up certain types of boundary conditions.

Suppose that all external forces such as the weight of the string and resistance forces, other than forces at the end points, can be neglected. Consider a segment of the string not containing an end point and whose projection onto the x axis has length Δx. Since x components of displacements are negligible, the mass of the segment is $\delta\,\Delta x$, where the constant δ is the mass per unit length of the string. At time t, the y component of the force exerted by the string on the segment at the left-hand end (x,y) is $V(x,t)$, given by equation (2). The tangential force **S** exerted on the other end of the segment by the part of the string to the right is also indicated in Fig. 1. Its y component $V(x + \Delta x, t)$ evidently satisfies the relation

$$\frac{V(x + \Delta x, t)}{H} = \tan\beta,$$

where β is the angle of inclination of the string at that other end of the segment. That is,

$$V(x + \Delta x, t) = Hy_x(x + \Delta x, t) \qquad (H > 0). \tag{3}$$

Note that, except for a minus sign, this is equation (2) when the argument x there is replaced by $x + \Delta x$.

Now the acceleration of the end (x,y) in the y direction is $y_{tt}(x,t)$. Consequently, by Newton's second law of motion (mass times acceleration equals force), it follows from equations (2) and (3) that

$$\delta\,\Delta x\, y_{tt}(x,t) = -Hy_x(x,t) + Hy_x(x + \Delta x, t), \tag{4}$$

approximately, when Δx is small. Hence

$$y_{tt}(x,t) = \frac{H}{\delta}\lim_{\Delta x \to 0}\frac{y_x(x + \Delta x, t) - y_x(x,t)}{\Delta x} = \frac{H}{\delta}\,y_{xx}(x,t)$$

whenever these partial derivatives exist.

Thus the function $y(x,t)$, which represents the transverse displacements in a stretched string under the conditions stated above, satisfies the one-dimensional *wave equation*

(5) $$y_{tt}(x,t) = a^2 y_{xx}(x,t) \qquad (a^2 = H/\delta).$$

The constant a has the physical dimensions of velocity.

One can choose units for the time variable so that $a = 1$ in the wave equation. More precisely, if we make the substitution $\tau = at$, the chain rule shows that

$$\frac{\partial y}{\partial t} = a\frac{\partial y}{\partial \tau} \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = a\frac{\partial}{\partial \tau}\left(a\frac{\partial y}{\partial \tau}\right) = a^2\frac{\partial^2 y}{\partial \tau^2}.$$

Equation (5) then becomes $y_{\tau\tau} = y_{xx}$.

4. MODIFICATIONS AND END CONDITIONS

When external forces parallel to the y axis act along the string, we let F denote the force per unit length of string, the positive sense of F being that of the y axis. Then a term $F\,\Delta x$ must be added on the right-hand side of equation ( ), Sec. 3, and the equation of motion is

(1) $$y_{tt}(x,t) = a^2 y_{xx}(x,t) + \frac{F}{\delta}.$$

In particular, with the y axis vertical and its positive sense upward, suppose that the external force consists of the weight of the string. Then $F\,\Delta x = -\delta\,\Delta x\, g$, where the positive constant g is the acceleration due to gravity; and equation (1) becomes the linear nonhomogeneous equation

(2) $$y_{tt}(x,t) = a^2 y_{xx}(x,t) - g.$$

In equation (1), F may be a function of x, t, y, or derivatives of y. If the external force per unit length is a damping force proportional to the velocity in the y direction, for example, F is replaced by $-By_t$, where the positive constant B is a damping coefficient. Then the equation of motion is linear and homogeneous:

(3) $$y_{tt}(x,t) = a^2 y_{xx}(x,t) - by_t(x,t) \qquad (b = B/\delta).$$

If an end $x = 0$ of the string is kept fixed at the origin at all times $t \geqslant 0$, the boundary condition there is clearly

(4) $$y(0,t) = 0 \qquad (t \geqslant 0).$$

But if that end is permitted to slide along the y axis and if the end is moved along that axis with a displacement $f(t)$, the boundary condition is the linear nonhomogeneous one

(5) $$y(0,t) = f(t) \qquad (t \geqslant 0).$$

Suppose that the left-hand end is attached to a ring which can slide along the y axis. When a force $F(t)\,(t > 0)$ in the y direction is applied to that end, $F(t)$ is