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# Introduction to the Mathematical Theory of Compressible Flow

Antonín Novotný  
and Ivan Straškraba

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# Introduction to the Mathematical Theory of Compressible Flow

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For Stania, Filip  
and Jakub  
For Jana

## PREFACE

The topic of this book is a mathematical description of Newtonian compressible fluids in the steady and unsteady regime. The history of attempts to describe rigorously the flow of a compressible fluid covered a long time beginning from observations of L. Euler in the middle of the 18th century and of C. Navier (Navier, 1827), H. Poisson (Poisson, 1829) and G. Stokes (Stokes, 1845) in the first half of the 19th century, and continue up to now. Despite the fact that the governing equations, called the Euler equations (in the inviscid case) and Navier–Stokes equations (in the viscous case), have been known for a very long time we are far from being satisfied with the completeness of their mathematical analysis (in both cases). Nevertheless the considerable effort of outstanding analysts cited throughout the book brought its fruits and a great number of nontrivial results for compressible fluids has been achieved. This book is an attempt to map the situation in the mathematical theory of compressible flow and present important and up-to-date results in a clear instructive form accessible to a wide audience despite the sophisticated techniques used to overcome modest a priori information derived directly from the equations.

We started with the realization of a rather different project including also a numerical treatment of the Euler and Navier–Stokes equations with M. Feistauer and J. Felcman around 1998. It soon appeared that the scope was too large for one monograph and so we accepted with great relief the proposal from Oxford University Press to split the book into two separate monographs, the first of which (Feistauer *et al.*, 2003) has already been published.

As already mentioned, the book covers Newtonian compressible fluids, more specifically Euler equations and Navier–Stokes equations in isentropic or barotropic regimes. We do not deal with heat conducting flows except for references to results for small data (that is under the assumption that appropriate norms of the given quantities are small enough). There is currently being published a research monograph by E. Feireisl (Feireisl, 2003a) devoted to this subject. There is a vast literature about different kinds of non-Newtonian fluids and we do not go into this business at all. So in this respect the present monograph covers only a part of the available mathematical results for compressible fluids.

We have adopted a textbook style. Even well-known basic theorems are recalled in the introductory chapter. This makes the book essentially selfcontained. There are sections called *heuristic approach*, where we describe the main ideas of proofs. These sections may be sufficient for an experienced reader to understand the subject without wasting time in numerous details. On the other hand, less experienced readers, nonspecialists and students will find in the book even standard technical details.

Let us briefly describe the contents. We start with the introductory Chapter 1,

where different models for compressible fluids are derived and some fundamental mathematical results are surveyed. The results are given without proofs but detailed references are given there.

Chapter 2 surveys the theoretical aspects of the Euler system for inviscid compressible fluids with the necessary background from the theory of hyperbolic conservation laws. Representative local and global existence results are proved in detail, relying on recent publications, to give the present state of affairs. The generality here is modest, and this is mostly due to the lack of results for the Euler equations.

Chapter 3 is preparatory for the subsequent treatment of Navier–Stokes systems. Some specific mathematical tools for these equations, adjusted especially to steady equations, are developed here. The proofs, unlike Chapter 1, are mostly given and only in a few cases are they cited.

Chapter 4 is devoted to the theory of weak solutions for steady Navier–Stokes systems for compressible fluids with large data (that is without the restriction described above as the assumption of small data) in the barotropic regime. A complete and detailed proof of the existence of weak solutions is given and modifications for unbounded and exterior domains as well as for different boundary conditions are discussed thoroughly. A survey of known results on this issue is given in the bibliographic remarks.

Chapter 5 concerns strong solutions of steady Navier–Stokes equations. The existence of regular solutions is proved, paid for by the assumption of small data. Note that *unconditional* regularity of solutions both for steady and nonsteady equations is not known.

Chapter 6 again collects advanced mathematical tools, now adjusted to nonsteady problems. This includes properties of abstract functions in Bochner spaces, commutators and the study of the (renormalized) equation of continuity.

Chapter 7 is mainly devoted to the weak existence theory for nonsteady Navier–Stokes equations in the barotropic regime. Again a discussion of different regions and boundary conditions is included.

In Chapter 8, the global behavior of solutions in time is investigated and the related equilibrium problem is described.

In the final Chapter 9, the existence of strong solutions for nonsteady Navier–Stokes equations is studied and available existence and uniqueness results are reviewed.

Chapter 2 dealing with the Euler equations is essentially self-contained and can be read independently of the other chapters. The same is true for Chapter 4 which deals with weak solutions of steady barotropic Navier–Stokes equations (and which requires only Chapter 3 to be exhaustive) and for Chapter 7 dealing with weak solutions of nonsteady barotropic Navier–Stokes equations (which needs only Chapter 3 and Section 4.4 of Chapter 4 to form a complete treatment). Each of Chapters 5 (about strong solutions for steady equations), 8 (about large time behavior of weak solutions) and 9 (about strong solutions in the nonsteady regime) are essentially self-contained as well. Also, we attempt to treat all



investigated systems in a uniform way taking into account their common nature.

We are grateful to E. Feireisl, G. P. Galdi, J. Heywood, P. Krejčí, V. Lovicar, J. Málek, S. Nazarov, J. Nečas, J. Neustupa, S. Novo, M. Padula, P. Penel, H. Petzeltová, K. Pileckas, M. Pokorný, M. Růžička, R. Salvi, A. Sequeira, A. Valli and A. Zlotnik, who are coauthors with at least one of us of several papers. We enjoyed working with them on more than one problem related to the subject.

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Carqueiranne and Prague,  
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A. N. & I. S.

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