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Harmonic Measure

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Harmonic Measure

During the last two decades several remarkable new results were discovered about harmonic measure in the complex plane. This book provides a survey of these results and an introduction to the branch of analysis that contains them. Many of these results, due to Bishop, Carleson, Jones, Makarov, Wolff, and others, appear here in book form for the first time.

The book is accessible to students who have completed standard graduate courses in real and complex analysis. The first four chapters provide the needed background material on univalent functions, potential theory, and extremal length, and each chapter has many exercises to further inform and teach the reader.

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To Dolores and Marianne

Preface

Several surprising new results about harmonic measure on plane domains have been proved during the last two decades. The most famous of these results are Makarov's theorems that harmonic measure on any simply connected domain is singular to Hausdorff measure Λ_α for all $\alpha > 1$ but absolutely continuous to Λ_α for all $\alpha < 1$. Also surprising was the extension by Jones and Wolff of Makarov's $\alpha > 1$ theorem to all plane domains. Further important new results include the work of Carleson, Jones and Wolff, and others on harmonic measure for complements of Cantor sets; the work by Carleson and Makarov, Bertilsson, Pommerenke, and others on Brennan's tantalizing conjecture that for univalent functions $\iint |\varphi'|^{2-p} dxdy < \infty$ if $\frac{4}{3} < p < 4$; several new geometric conditions that guarantee the existence of angular derivatives; and the Jones square sum characterization of subsets of rectifiable curves and its applications by Bishop and Jones to a variety of problems in function theory.

We wrote this book to explain these exciting new results and to provide beginning students with an introduction to this part of mathematics. We have tried to make the subject accessible to students who have completed graduate courses in real analysis from Folland [1984] or Wheeden and Zygmund [1977], for example, and in complex analysis from Ahlfors [1979] or Gamelin [2001], for example.

The first four chapters, along with the appendices on Hardy spaces, Hausdorff measures and martingales, provide a foundation that every student of function theory will need. In Chapter I we solve the Dirichlet problem on the half-plane and the disc and then on any simply connected Jordan domain by using the Carathéodory theorem on boundary continuity. Chapter I also includes brief introductions to hyperbolic geometry and univalent function theory. In Chapter II we solve the Dirichlet problem on domains bounded by finitely many Jordan curves and study the connection between the smoothness of a domain's boundary and the smoothness of its Poisson kernel. Here the main tools are two classical theorems about conjugate functions. Chapter II and the discussion in Chapter III of Wiener's solution of the Dirichlet problem on arbitrary domains follow the 1985 UCLA lecture course by Carleson. The introduction to extremal length in Chapter IV is based on the Institut Mittag–Leffler lectures of Beurling [1989]. Chapter V contains some applications of extremal length, such as Teichmüller's Modulsatz and some newer theorems about angular derivatives, that are not found in other books. Chapter VI is a blend of the classical theorems of F. and M. Riesz, Privalov, and Plessner and the more recent theorems of McMillan, Makarov, and Pommerenke on the comparison of harmonic measure and one dimensional Hausdorff measure for simply connected domains. Chapter VII surveys the beautiful circle of ideas around Bloch functions, univalent functions, quasicircles, and A^p weights. Chapter VIII is an exposition of Makarov's deeper results on the relations between harmonic measure in simply connected domains and Hausdorff measures and the work of Carleson and Makarov concerning Brennan's conjecture. Chapter IX discusses harmonic measure on infinitely connected plane domains. Chapter X begins by introducing the Lusin area function, the Schwarzian derivative, and the Jones square sums, and then applies these ideas to several problems about univalent functions and harmonic measures. The thirteen appendices at the end of the text provide further related material.

For space reasons we have not treated some important related topics. These include the connections between Chapters VIII and IX and thermodynamical formalism and several other connections between complex dynamics and harmonic measure. We have emphasized Wiener's solution of the Dirichlet problem instead of the Perron method. The beautiful Perron method can be found in Ahlfors [1973] and Tsuji [1959]. We also taken a few detours around the theory of prime ends. There are excellent discussions of prime ends in Ahlfors [1973], Pommerenke [1975], and Tsuji [1959]. Finally, the theory of harmonic measure in higher dimensions has a different character, and we have omitted it entirely.

At the end of each chapter there is a brief section of biographical notes and a section called "Exercises and Further Results". An exercise consisting of a stated result without a reference is meant to be homework for the reader. "Further results" are outlines, with detailed references, of theorems not in the text.

Results are numbered lexicographically within each chapter, so that Theorem 2.4 is the fourth item in Section 2 of the same chapter, while Theorem III.2.4 is from Section 2 of Chapter III. The same convention is used for formulas, so that (3.2) is in the same chapter, while (IV.6.4) refers to (6.4) from Chapter IV.

Many of the results that inspired us to write this book are also covered in

Pommerenke's excellent book [1991]. However, our emphasis differs from the one in Pommerenke [1991] and we hope the two books will complement each other.

Some unpublished lecture notes from a 1986 Nachdiplom Lecture course at Eidgens̈sische Technische Hochschule Zurich by the first listed author and the out-of-print monograph Garnett [1986] were preliminary versions of the present book.

The web page

http://www.math.washington.edu/~marshall/HMcorrections.html

will list corrections to the book. Though we have tried to avoid errors, the observant reader will no doubt find some. We would appreciate receiving email at marshall@math.washington.edu about any errors you come across.

Many colleagues, friends, and students have helped with their comments and suggestions. Among these, we particularly thank A. Baernstein, M. Benedicks, D. Bertilsson, C. J. Bishop, K. Burdzy, L. Carleson, S. Choi, R. Chow, M. Essèn, R. Gundy, P. Haissinski, J. Handy, P. Jones, P. Koosis, N. Makarov, P. Mateos, M. O'Neill, K. Øyma, R. Pérez-Marco, P. Poggi-Corridini, S. Rohde, I. Uriarte-Tuero, J. Verdera, S. Yang and S. Yoshinobu.

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Los Angeles and Seattle — John B. Garnett

Seattle and Bergen — Donald E. Marshall

Contents

I
Jordan Domains

To begin we construct harmonic measure and solve the Dirichlet problem in the upper half-plane and the unit disc. We next prove the Fatou theorem on nontangential limits. Then we construct harmonic measure on domains bounded by Jordan curves, via the Riemann mapping theorem and the Carathéodory theorem on boundary correspondence. We review two topics from classical complex analysis, the hyperbolic metric and the elementary distortion theory for univalent functions. We conclude the chapter with the theorem of Hayman and Wu on lengths of level sets. Its proof is an elementary application of harmonic measure and the hyperbolic metric.

1. The Half-Plane and the Disc

Write $\mathbb{H} = \{z : \text{Im}z > 0\}$ for the upper half-plane and $\mathbb{R}$ for the real line. Suppose $a < b$ are real. Then the function

$$\theta = \theta(z) = \arg\left(\frac{z-b}{z-a}\right) = \text{Im}\log\left(\frac{z-b}{z-a}\right)$$

is harmonic on $\mathbb{H}$, and $\theta = \pi$ on (a, b) and $\theta = 0$ on $\mathbb{R} \setminus [a, b]$.

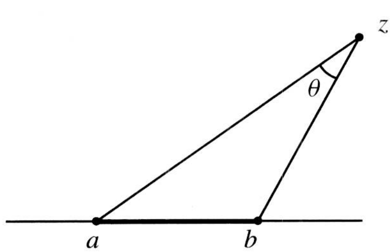

Figure I.1 The harmonic function $\theta(z)$.

Viewed geometrically, $\theta(z) = \text{Re}\varphi(z)$ where $\varphi(z)$ is any conformal mapping from $\mathbb{H}$ to the strip $\{0 < \text{Re}z < \pi\}$ which maps (a, b) onto $\{\text{Re}z = \pi\}$ and $\mathbb{R} \setminus [a, b]$ into $\{\text{Re}z = 0\}$. Let $E \subset \mathbb{R}$ be a finite union of open intervals and write $E = \bigcup_{j=1}^{n}(a_j, b_j)$ with $b_{j-1} < a_j < b_j$. Set

$$\theta_j = \theta_j(z) = \arg\left(\frac{z - b_j}{z - a_j}\right)$$

and define the **harmonic measure** of E at $z \in \mathbb{H}$ to be

$$\omega(z, E, \mathbb{H}) = \sum_{j=1}^{n} \frac{\theta_j}{\pi}. \tag{1.1}$$

Then

(i) $0 < \omega(z, E, \mathbb{H}) < 1$ for $z \in \mathbb{H}$,
(ii) $\omega(z, E, \mathbb{H}) \to 1$ as $z \to E$, and
(iii) $\omega(z, E, \mathbb{H}) \to 0$ as $z \to \mathbb{R} \setminus \overline{E}$.

The function $\omega(z, E, \mathbb{H})$ is the unique harmonic function on $\mathbb{H}$ that satisfies (i), (ii), and (iii). The uniqueness of $\omega(z, E, \mathbb{H})$ is a consequence of the following lemma, known as **Lindelöf's maximum principle.**

Lemma 1.1 (Lindelöf). *Suppose the function $u(z)$ is harmonic and bounded above on a region Ω such that $\overline{\Omega} \neq \mathbb{C}$. Let F be a finite subset of $\partial\Omega$ and suppose*

$$\limsup_{z \to \zeta} u(z) \leq 0 \tag{1.2}$$

for all $\zeta \in \partial\Omega \setminus F$. Then $u(z) \leq 0$ on Ω.

Proof. Fix $z_0 \notin \overline{\Omega}$. Then the map $1/(z - z_0)$ transforms Ω into a bounded region, and thus we may assume Ω is bounded. If (1.2) holds for all $\zeta \in \partial\Omega$, then the lemma is the ordinary maximum principle. Write $F = \{\zeta_1, \dots, \zeta_N\}$, let $\varepsilon > 0$, and set

$$u_\varepsilon(z) = u(z) - \varepsilon \sum_{j=1}^{N} \log\left(\frac{\text{diam}(\Omega)}{|z - \zeta_j|}\right).$$

Then u_ε is harmonic on Ω and $\limsup_{z\to\zeta} u_\varepsilon(z) \leq 0$ for all $\zeta \in \partial\Omega$. Therefore $u_\varepsilon \leq 0$ for all ε, and

$$u(z) \leq \lim_{\varepsilon \to 0} \varepsilon \sum_{j=1}^{N} \log\left(\frac{\text{diam}(\Omega)}{|z - \zeta_j|}\right) = 0.$$

■

Lindelöf [1915] proved Lemma 1.1 under the weaker hypothesis that $\partial\Omega$ is infinite. See also Ahlfors [1973]. Exercise 3 and Exercise II.3 tell more about Lindelöf's maximum principle.

Given a domain Ω and a function $f \in C(\partial\Omega)$, the **Dirichlet problem** for f on Ω is to find a function $u \in C(\overline{\Omega})$ such that $\Delta u = 0$ on Ω and $u|_{\partial\Omega} = f$. Theorem 1.2 treats the Dirichlet problem on the upper half-plane $\mathbb{H}$.

Theorem 1.2. *Suppose $f \in C(\mathbb{R} \cup \{\infty\})$. Then there exists a unique function $u = u_f \in C(\overline{\mathbb{H} \cup \{\infty\}})$ such that u is harmonic on $\mathbb{H}$ and $u|_{\partial\mathbb{H}} = f$.*

Proof. We can assume f is real valued and $f(\infty) = 0$. For $\varepsilon > 0$, take disjoint open intervals $I_j = (t_j, t_{j+1})$ and real constants c_j, $j = 1, \dots, n$, so that the step function

$$f_\varepsilon(t) = \sum_{j=1}^{n} c_j \chi_{I_j}$$

satisfies

$$\| f_\varepsilon - f \|_{L^\infty(\mathbb{R})} < \varepsilon. \tag{1.3}$$

Set

$$u_\varepsilon(z) = \sum_{j=1}^{n} c_j \omega(z, I_j, \mathbb{H}).$$

If $t \in \mathbb{R} \setminus \bigcup \partial I_j$, then

$$\lim_{\mathbb{H} \ni z \to t} u_\varepsilon(z) = f_\varepsilon(t)$$

by (ii) and (iii). Therefore by (1.3) and Lemma 1.1,

$$\sup_{\mathbb{H}} \left| u_{\varepsilon_1}(z) - u_{\varepsilon_2}(z) \right| < \varepsilon_1 + \varepsilon_2.$$

Consequently the limit

$$u(z) \equiv \lim_{\varepsilon \to 0} u_\varepsilon(z)$$

exists, and the limit $u(z)$ is harmonic on $\mathbb{H}$ and satisfies

$$\sup_{\mathbb{H}} |u(z) - u_\varepsilon(z)| \le 2\varepsilon.$$

We claim that

$$\limsup_{z \to t} |u_\varepsilon(z) - f(t)| \le \varepsilon \tag{1.4}$$

for all $t \in \mathbb{R}$. It is clear that (1.4) holds when $t \notin \bigcup \partial I_j$. To verify (1.4) at the endpoint $t_{j+1} \in \partial I_j \cap \partial I_{j+1}$, notice that by (ii), (iii), and Lemma 1.1,

$$\sup_{\mathbb{H}} \left| c_j \omega(z, I_j, \mathbb{H}) + c_{j+1} \omega(z, I_{j+1}, \mathbb{H}) - \left(\frac{c_j + c_{j+1}}{2} \right) \omega(z, I_j \cup I_{j+1}, \mathbb{H}) \right| \leq \left| \frac{c_j - c_{j+1}}{2} \right|,$$

while

$$\lim_{z \to t_{j+1}} \left(\frac{c_j + c_{j+1}}{2} \right) \omega(z, I_j \cup I_{j+1}, \mathbb{H}) = \frac{c_j + c_{j+1}}{2}.$$

Hence all limit values of $u_\varepsilon(z)$ at t_{j+1} lie in the closed interval with endpoints c_j and c_{j+1}, and then (1.3) yields (1.4) for the endpoint t_{j+1}.

Now let $t \in \mathbb{R}$. By (1.4)

$$\limsup_{z \to t} |u(z) - f(t)| \leq \sup_{z \in \mathbb{H}} |u(z) - u_\varepsilon(z)| + \limsup_{z \to t} |u_\varepsilon(z) - f(t)| \leq 3\varepsilon.$$

The same estimate holds if $t = \infty$. Therefore u extends to be continuous on $\overline{\mathbb{H}}$ and $u|_{\partial \mathbb{H}} = f$. The uniqueness of u follows immediately from the maximum principle. ■

For $a < b$, elementary calculus gives

$$\omega(x + iy, (a, b), \mathbb{H}) = \frac{1}{\pi} \left(\tan^{-1}\left(\frac{x - a}{y}\right) - \tan^{-1}\left(\frac{x - b}{y}\right) \right) = \int_a^b \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}.$$

If $E \subset \mathbb{R}$ is measurable, we define the **harmonic measure** of E at $z \in \mathbb{H}$ to be

$$\omega(z, E, \mathbb{H}) = \int_E \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}. \tag{1.5}$$

When E is a finite union of open intervals this definition (1.5) is the same as definition (1.1). For $z = x + iy \in \mathbb{H}$, the density

$$P_z(t) = \frac{1}{\pi} \frac{y}{(x - t)^2 + y^2}$$

is called the **Poisson kernel** for $\mathbb{H}$. If $f \in C(\mathbb{R} \cup \{\infty\})$, the proof of Theorem 1.2 shows that

$$u_f(z) = \int_{\mathbb{R}} f(t) P_z(t) dt,$$

and for this reason u_f is also called the **Poisson integral** of f.