Mathematics and Its Applications

V. I. Gorbachuk and M. L. Gorbachuk

Boundary Value Problems for Operator Differential Equations

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by

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Boundary Value Problems for Operator Differential Equations

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Volume 48

SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be able to do something with it.

O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded non-sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and non-linearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the non-linear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, p-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five subseries: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

A differential equation of the form $y''(t)+ A(t)y(t) = 0$ looks very familiar and certainly a great many volumes have been written about the corresponding boundary-value problems. In this book, however, the equation above is an **operator** equation, and that makes it unique. The spectral analysis of Sturm-Liouville differential operator equations in the case of an infinite-dimensional space, began very recently (in spite of the many potential applications) and this book, by two well known researchers in the area, aims to present the subject systematically together with its natural links to the important area of extensions of symmetric operators.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Bussum, January 1990

Michiel Hazewinkel

Preface

The book deals with the theory of boundary value problems for second-order operator differential equations of the form

$$y''(t) + A(t)y(t) = 0 \qquad (t \in [a, b], \quad -\infty < a < b < \infty),$$

where the $A(t)$ are semi-bounded self-adjoint operators on a separable Hilbert space $\mathfrak{H}$. The study of differential equations whose coefficients are unbounded operators on a Hilbert or Banach space is useful not only because these include many partial differential equations but also because it offers the possibility of looking at ordinary as well as partial differential operators from a single viewpoint.

The studies of the last 30 years have enriched the theory of operator differential equations with significant results. The presentation of the Gauchy problem and the stability theory of solutions can be found both in textbooks on the theory of operators (Hille–Phillips [1], for example) and in special monographs (see, for example, Lions [1], S. Krein [1], Daletsky–M. Krein [1]). The spectral analysis of the Sturm–Liouville operator differential equation, which was given a lot of attention in the scalar case and in the case of a finite-dimensional $\mathfrak{H}$, began its development quite recently in the case of an infinite-dimensional space and an unbounded operator potential $A(t)$. Naturally, then, there are no books which reflect on this trend. In this book we would like to fill this gap, if not completely, then at least partially.

For the scalar Sturm–Liouville equations one usually considers two cases, that of a bounded and that of an unbounded interval, i.e. the regular and the singular case. They are known to differ as regards formulation of problems, methods of investigation, and fields of applications.

When studying operator equations one must take into account not only boundedness or unboundedness of the interval, but also the character of unboundedness of the potential. The fact whether the operators $A(t)$ are lower or upper semi-bounded proved to be fundamentally important. In this connection, the equations are divided into elliptic ($A(t) \leq 0$) and hyperbolic ($A(t) \geq 0$). The Laplace equation and the D'Alembert equation serve as respective models for these. In view of the limited volume of the book and the unlimited stream of results we mainly consider the case of a bounded interval and present the theory of dissipative (in particular, self-adjoint) boundary value problems. It is quite natural that while selecting material the authors' personal interests somewhat prevailed.

In the first chapter we give basic definitions and (almost without proofs) classical theorems from the theory of Banach, Hilbert, and locally convex topological spaces, and from the theory of linear operators on them. This information is necessary for understanding the subsequent chapters. Since the principal object of study of this book is a vector-function with values in infinite-dimensional spaces, while the major instrument of investigation is the operational calculus of self-adjoint operators, these are given greater attention.

The second chapter deals with the theory of boundary values of solutions of second-order elliptic operator differential equations which are smooth inside the interval. On the one hand, this theory plays an important role in the formulation and investigation of boundary value problems for such equations; on the other hand, it gives a uniform approach to the theory of boundary values of analytic functions, allowing one to obtain, in particular, well-known theorems concerning existence of boundary values of harmonic (analytic) functions $u(x,t)$ in the upper half-plane, in the Schwartz space of distributions if $u(x,t)$ has power growth as t approaches the real axis and in the spaces of ultradistributions if $u(x,t)$ has exponential growth as t approaches the real axis. This theory also makes it possible to establish analogous results for solutions of homogeneous partial differential equations different from the Laplace equation which are smooth inside the domain.

The proofs of the principal results are based on the spectral representation theorem for a self-adjoint operator on a Hilbert space. Also, chains of spaces with positive and negative norms and their inductive and projective limits are essentially used. Their theory is set forth in sufficient detail.

The third chapter consists in fact of two parts. The first part is devoted to the theory of extensions of abstract symmetric operators. Its presentation somewhat differs from the traditional one and is adapted to the theory of boundary value problems. The description of various classes of extensions (maximal dissipative, self-adjoint, solvable and others), as well as the structure of the spectrum of extensions from these classes, is given in terms of so-called boundary value spaces. The latter are convenient and natural because they turn into the usual boundary condition in certain concrete situations. Here, an important place is occupied by theorems about various representations of binary relations in a Hilbert space. These are the starting point in constructing the theory of extensions.

In the second part this theory is applied to investigating boundary value problems for the formally self-adjoint Sturm–Liouville expression with operator potential of hyperbolic type given on a bounded interval. The minimal operator generated by it is symmetric and has infinite deficiency numbers when $\dim \mathfrak{H} = \infty$. Each extension of it is associated with some boundary value problem in the sense that vector-functions in the domain of the extension satisfy a definite boundary condition at the ends of the interval. Therefore, a lot of properties of extensions (self-adjointness, maximal dissipativeness, structure of the spectrum, etc.) can be completely described in terms of the coefficients of the equation and the boundary conditions corresponding to these extensions.

The fourth chapter contains results concerning the spectral theory of boundary value problems in the elliptic case. The various classes of dissipative problems are described in it, and the asymptotic distribution of their eigenvalues is studied. Particular attention is paid to self-adjoint boundary value problems with discrete spectrum. The behaviour of the distribution function of the eigenvalues of such a problem depends essentially on the boundary conditions. Classes of self-adjoint boundary value problems for which the dominant term in the asymptotics of the distribution function has given order of growth are singled out. For some of them the second-order terms of the asymptotics are studied, and an estimate of the remainder is given. We also establish a connection between the asymptotic behaviour of the distribution function of the eigenvalues and the smoothness up to the boundary of elements in the domain of the self-adjoint extension corresponding to the boundary value problem considered.

The fifth, and last, chapter deals with the theory of boundary values at zero of solutions of a first-order differential equation of the form $y'(t) + Ay(t) = 0$ $(t \in (0, \infty))$ in a Banach space. One of the reasons to construct such a theory is the hope to find a general approach to the well-known Riesz theorems concerning boundary values in L_p spaces $(p \neq 2)$ of analytic functions, from the viewpoint of evolution equations.

Since infinity as well as zero is a singular point for such an equation, we also discuss results related to the behaviour of solutions at ∞, which is related to stability theory.

The book provides a number of examples which prove that the operator approach makes it possible not only to extend the class of already studied partial differential equations and their boundary value problems, but also to look from another point of view to the spectral theory of self-adjoint boundary value problems for such classical expressions as those of Laplace and D'Alembert.

We will not always formulate the results in the most general form. We have preferred to select a somewhat average level of generality (a "golden mean"). The rest is added by way of comments and references.

To make reading easy, the principal statements are distinguished in the form of theorems, lemmas, corollaries, and remarks as well as formulas.

It gives us pleasure to thank M.G. Krein and Ju.M. Berezansky, whose great influence we felt throughout our scientific activities. Their work on the theory of boundary value problems and our continual contact with them determined the subject of our investigations and the subject matter of the book.

In writing the manuscript we were helped by our pupils and colleagues. Sections 1–3 (Chapter 3) were written together with A. N. Kochubei, Section 6 (Chapter 1) and Section 3 (Chapter 4) - with V. A. Mikhailets, Section 6 (Chapter 4) - with L. I. Vainerman. V. V. Gorodetsky, A. I. Kashpirovski, A. V. Knyaziuk, V. V. Levchuk, L. B. Fedorova participated in the discussion of some sections of the book. We sincerely thank all of them.

In preparing the present version we were helped very much by A. N. Kochubei and we would like to express our particular gratitude to him.

Contents

CHAPTER 1

Some Information from the Theory of Linear Operators

1. Banach Spaces and Continuous Linear Operators on Them

1.1. NORMED AND BANACH SPACES

A set $\mathfrak{X}$ is called a complex normed space if

(1) $\mathfrak{X}$ is a vector space over the field $\mathbb{C}^1$ of complex numbers;

(2) for each element $x \in \mathfrak{X}$ there is defined a non-negative number $\|x\|$ (the norm of x) possessing the following properties:

(i) $\|\alpha x\| = |\alpha| \, \|x\| \quad (\forall x \in \mathfrak{X}, \forall \alpha \in \mathbb{C}^1)$;

(ii) $\|x + y\| \le \|x\| + \|y\| \quad (\forall x, y \in \mathfrak{X})$;

(iii) $\|x\| = 0$ if and only if $x = 0$.

A sequence $x_n \in \mathfrak{X}$ $(n \in \mathbb{N} = \{1, 2, \dots\})$ is said to converge in $\mathfrak{X}$ to an element x if $\lim_{n\to\infty} \|x_n - x\| = 0$. A sequence x_n $(n \in \mathbb{N})$ from $\mathfrak{X}$ is called fundamental if $\lim_{m,n\to\infty} \|x_n - x_m\| = 0$.

A normed space $\mathfrak{B}$ is called a Banach space if it is complete, i.e. each fundamental sequence in it has a limit. Any incomplete normed space can be completed, i.e. imbedded into a certain Banach space as a dense linear subset.

A Banach space is called separable if it contains a countable dense set. We will consider only separable Banach spaces.

If another norm $\|x\|_2$ is given in the normed space $\mathfrak{X}$ with norm $\|x\|_1$, then for convergence with respect to $\|\cdot\|_2$ of each sequence that is convergent with respect to $\|\cdot\|_1$ it is necessary and sufficient that there exists a positive constant c such that

$$\|x\|_2 \le c\|x\|_1.$$

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called topologically equivalent if convergence with respect to one of them implies convergence with respect to the other. It is clear that in a normed space $\mathfrak{X}$, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are topologically equivalent if and only if there exist positive constants c_1, c_2 such that

$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1. \tag{1.1}$$

1.2. ALGEBRAS, BANACH ALGEBRAS, IDEALS

A complex linear space $\mathfrak{A}$ over the field $\mathbb{C}^1$ is called an algebra if a multiplication of its elements is introduced in such a way that the multiplication operation is distributive relative to addition, and commutes with multiplication by complex numbers. If multiplication is commutative, i.e. $xy = yx$ $(\forall x, y \in \mathfrak{A})$, then the algebra is called commutative.

If the algebra $\mathfrak{A}$ is a Banach space and multiplication is continuous with respect to each of the factors, then $\mathfrak{A}$ is called a Banach algebra. A unit element in $\mathfrak{A}$ is a vector $e \in \mathfrak{A}$ such that $xe = ex = x$ for each $x \in \mathfrak{A}$. An algebra containing a unit element is called an algebra with a unit. If an algebra has no unit it can always be complemented so that the extension contains a unit. In any Banach algebra with a unit one can replace the norm by an equivalent one for which the following properties are satisfied:

$$\|xy\| \leq \|x\|\|y\| \quad (\forall x, y \in \mathfrak{A}); \quad \|e\| = 1.$$

Therefore, it is usually assumed that a Banach algebra has a unit, and the above-mentioned properties for a norm hold.

A set I of elements from the algebra $\mathfrak{A}$ is called a (two-sided) ideal if:

(i) $x + y \in I$ for any $x, y \in I$;

(ii) $xz \in I$, $zx \in I$ for any $x \in I$, any $z \in \mathfrak{A}$.

An ideal I of $\mathfrak{A}$ is called proper if $I \neq \mathfrak{A}$. A maximal ideal is a proper ideal which is not contained in other proper ideals.

1.3. CONCRETE BANACH SPACES

The space $\mathbb{C}^n$ consists of all ordered n-tuples $\{\alpha_1, \cdots, \alpha_n\}$ of complex numbers. It is an n-dimensional Banach space with respect to the norm

$$\|\alpha\|_{\mathbb{C}^n} = \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{\frac{1}{2}}.$$

We denote by $\mathbb{R}^n$ its subset of all n-tuples of real numbers.

The space $C^k[a, b]$ is the set of all k times continuously-differentiable complex-valued functions on the closed interval $[a, b]$ of the real line $\mathbb{R}^1$. The norm of a function $f \in C^k[a, b]$ is defined by

$$\|f\|_{C^k[a,b]} = \max_{t \in [a,b]} \{|f(t)|, \ldots, |f^{(k)}(t)|\}.$$

With this norm, $C^k[a, b]$ forms a Banach space as well as a Banach algebra. Let $C^0[a, b] = C[a, b]$. It is the space of all continuous functions $f(t)$ $(a \leq t \leq b)$, and

$$\|f\|_{C[a,b]} = \max_{t \in [a,b]} |f(t)|.$$

The space $L_p(a, b)$. The elements of this space are the complex-valued functions

f on (a,b) for which

$$\|f\|_{L_p(a,b)} = \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty \tag{1.2}$$

(functions coinciding almost everywhere are identified). $L_p(a,b)$ with the norm (1.2) is a Banach space. The space $L_p(\Omega)$, where Ω is a domain in $\mathbb{R}^n$, is similarly defined.

1.4. BOUNDED LINEAR OPERATORS ON A BANACH SPACE

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be Banach spaces. A mapping $A : \mathfrak{B}_1 \to \mathfrak{B}_2$ is called a linear operator if

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \qquad (\forall \alpha, \beta \in \mathbb{C}^1).$$

The linear operator A is continuous if $x_n \to x$ in $\mathfrak{B}_1$ implies $Ax_n \to Ax$ $(n \to \infty)$ in $\mathfrak{B}_2$, and it is called bounded if

$$\|Ax\|_{\mathfrak{B}_2} \le c\|x\|_{\mathfrak{B}_1} \tag{1.3}$$

for some positive constant c. The least of the constants c in the inequality (1.3) is the norm of the operator A:

$$\|A\| = \sup_{x \in \mathfrak{B}_1} \frac{\|Ax\|_{\mathfrak{B}_2}}{\|x\|_{\mathfrak{B}_1}} = \sup_{x : \|x\|_{\mathfrak{B}_1} = 1} \|Ax\|_{\mathfrak{B}_2}. \tag{1.4}$$

It is not difficult to show that a linear operator A is continuous if and only if it is bounded.

The set of all bounded linear operators $A : \mathfrak{B}_1 \to \mathfrak{B}_2$ is denoted by $[\mathfrak{B}_1, \mathfrak{B}_2]$. This set is a Banach space with the norm (1.4) and the natural definition of addition of operators and multiplication of an operator by a number:

$$(A + B)x = Ax + Bx, \qquad (\alpha A)x = A(\alpha x) = \alpha Ax.$$

If $B : \mathfrak{B}_2 \to \mathfrak{B}_3$, then the formula

$$Cx = B(Ax) \qquad (x \in \mathfrak{B}_1)$$

defines a linear operator $C : \mathfrak{B}_1 \to \mathfrak{B}_3$, called the product of A and B. For bounded operators A, B the operator C is bounded too, and

$$\|C\| \le \|A\| \|B\|.$$

The inverse of A is the operator $A^{-1} : \mathfrak{B}_2 \to \mathfrak{B}_1$ for which

$$A^{-1}A = E_1, \qquad AA^{-1} = E_2,$$

where $E_1 : \mathfrak{B}_1 \to \mathfrak{B}_1$ and $E_2 : \mathfrak{B}_2 \to \mathfrak{B}_2$ are the identity operators on $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively.

The set of bounded linear operators from $\mathfrak{B}$ into $\mathfrak{B}$ is denoted by $[\mathfrak{B}]$, i.e. $[\mathfrak{B}] = [\mathfrak{B}, \mathfrak{B}]$. With the above-defined multiplication of operators, $[\mathfrak{B}]$ forms a Banach algebra.

Two Banach spaces are called isomorphic if there exists a one-to-one continuous linear mapping of one space onto another which has a continuous inverse. Such a mapping is called an isomorphism. A norm-preserving isomorphism is called an isometric isomorphism.

Let us formulate some important theorems.

THEOREM 1.1. (Banach). *Assume that an operator $A \in [\mathfrak{B}_1, \mathfrak{B}_2]$ maps a Banach space $\mathfrak{B}_1$ onto a Banach space $\mathfrak{B}_2$ one-to-one. Then $A^{-1} \in [\mathfrak{B}_2, \mathfrak{B}_1]$, hence the spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are isomorphic.*

It follows from this theorem that if a space $\mathfrak{B}$ is Banach with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$, then for the topological equivalence of these norms the realization of any one-side bound in (1.1) is sufficient.

THEOREM 1.2. (Banach–Steinhaus). *Let $\mathfrak{M}$ be the set of operators from $[\mathfrak{B}_1, \mathfrak{B}_2]$ such that*

$$\sup_{A\in\mathfrak{M}} \|Ax\| < \infty$$

for each $x \in \mathfrak{B}_1$. Then the set $\mathfrak{M}$ is bounded, that is

$$\sup_{A\in\mathfrak{M}} \|A\| < \infty.$$

This theorem is also called the uniform boundedness principle. It follows from this theorem that if a sequence of operators $A_n \in [\mathfrak{B}_1, \mathfrak{B}_2]$ converges on each element $x \in \mathfrak{B}_1$, then the inequality

$$Ax = \lim_{n\to\infty} A_n x$$

defines a continuous linear operator from $\mathfrak{B}_1$ into $\mathfrak{B}_2$.

A linear operator $A : \mathfrak{B}_1 \to \mathfrak{B}_2$ is called compact, or completely continuous, if it maps any bounded set of $\mathfrak{B}_1$ to a compact set of $\mathfrak{B}_2$. A compact operator is continuous. A linear operator is compact if it maps the unit sphere of $\mathfrak{B}_1$ into a compact set of $\mathfrak{B}_2$.

1.5. DIRECT SUMS OF SUBSPACES AND PROJECTORS

A closed linear subset of a Banach space $\mathfrak{B}$ is called a subspace of it. A Banach space is said to be the direct sum of its subspaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$, which is written as

$$\mathfrak{B} = \mathfrak{B}_1 \dot{+} \mathfrak{B}_2, \tag{1.5}$$

if an arbitrary element $x \in \mathfrak{B}$ can be uniquely represented in the form

$$x = x_1 + x_2 \qquad (x_i \in \mathfrak{B}_i). \tag{1.6}$$

The direct decomposition (1.5) generates two linear operators $P_1 : \mathfrak{B} \to \mathfrak{B}_1$ and $P_2 : \mathfrak{B} \to \mathfrak{B}_2$, defined as follows: if x is represented in the form (1.6), then

$$P_1 x = x_1, \quad P_2 x = x_2.$$

The following properties of the operators $P_i \quad (i = 1, 2)$ are trivial:

$$P_i^2 = P_i, \qquad P_1 + P_2 = E, \qquad P_1 P_2 = P_2 P_1 = 0$$

(E is the identity operator). Moreover, the P_i are continuous. In fact, introduce on $\mathfrak{B}$ a new norm

$$\|x\|_1 = \|x_1\| + \|x_2\|.$$

Since $x = x_1 + x_2$, $\|x\| \le \|x_1\| + \|x_2\| = \|x\|_1$. If a sequence $x^n = x_1^n + x_2^n$ is fundamental with respect to the norm $\|\cdot\|_1$, then the sequences x_1^n and x_2^n are fundamental. So, $x_1^n \to x_1 \in \mathfrak{B}_1$, $x_2^n \to x_2 \in \mathfrak{B}_2$. Then $x^n \to x_1 + x_2 \in \mathfrak{B}$ in both norms $\|\cdot\|$ and $\|\cdot\|_1$. Therefore, $\mathfrak{B}$ is also complete under the norm $\|\cdot\|$. Hence there exists a constant $c > 0$ such that

$$\|x\|_1 \le c\|x\|,$$

whence

$$\|P_i x\| = \|x_i\| \le \|x\|_1 \le c\|x\| \qquad (i = 1, 2),$$

i.e. the operators P_i are continuous.

An operator $P \in [\mathfrak{B}]$ is called a projection operator or projector if $P^2 = P$. Thus, the operators P_1, P_2, constructed above, are projectors. The converse assertion is also valid. If P_1, P_2 are projectors on $\mathfrak{B}$ and $P_1 + P_2 = E$, then they generate a direct decomposition (1.5), where $\mathfrak{B}_1 = P_1\mathfrak{B}$, $\mathfrak{B}_2 = P_2\mathfrak{B}$.

A decomposition of $\mathfrak{B}$ into a direct sum of several subspaces is defined in the natural way.

1.6. THE DIRECT SUM OF BANACH SPACES

Let $\mathfrak{B}_1, \cdots, \mathfrak{B}_n$ be Banach spaces with norms $\|\cdot\|_1, \cdots, \|\cdot\|_n$, respectively. By the direct sum

$$\mathfrak{B} = \mathfrak{B}_1 \dotplus \mathfrak{B}_2 \dotplus \cdots \dotplus \mathfrak{B}_n$$

we mean the Banach space of ordered n-tuples $x = \{x_1, x_2, \cdots, x_n\}$ $(x_i \in \mathfrak{B}_i)$ with the natural definition of addition of n-tuples and multiplication of an n-tuple by a number. The norm in $\mathfrak{B}$ is given by

$$\|x\| = \left(\sum_{i=1}^{n} \|x_i\|_i^2 \right)^{\frac{1}{2}}.$$