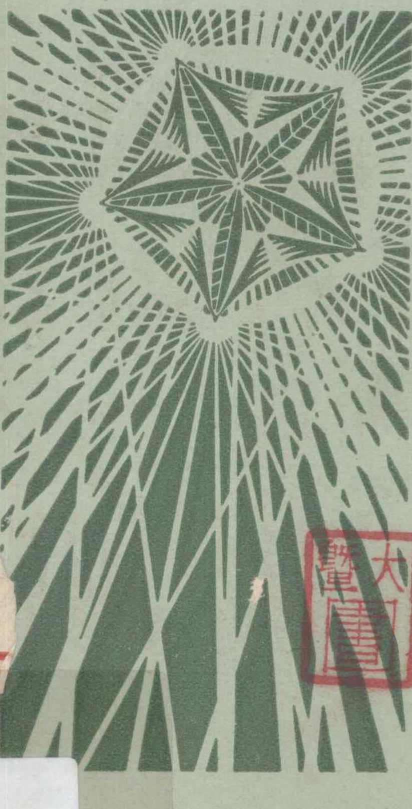
GEOMETRY OF AGGREGATES



I. W. VAN SPIEGEL

I. W. VAN SPIEGEL, DR. MATH.

GEOMETRY OF AGGREGATES

VOL GOE DE COURAGIE

ASSEN MCMLVII

VAN GORCUM & COMP. N.V. - G. A. HAK & DR. H. J. PRAKKE

The cover of the book, derived from the birational representation of the surface of order three by a plane as is treated in chapter V, has been designed by H. F. A. Veenker

Printed in the Netherlands by Royal VanGorcum Ltd., Assen

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INTRODUCTION

In the year 1881 F. Schur (1) published a paper with the title: "Ueber die durch collineare Grundgebilde erzeugten Curven und Flächen", in which he considered some linear systems of projectively related pencils, sheaves and spaces of planes in three-dimensional projective space, and the manifolds connected with these systems. This investigation was more systematically continued by Th. Reye (1) in an extensive series of publications, "Ueber lineare Mannigfaltigkeiten projectiver Ebenenbüschel und collinearer Bündel oder Räume". One of the consequences of this systematical research was a large number of problems, not all of which have been solved to-day. Like F. Schur, Th. Reye obtained his results in a purely synthetic way. Both investigators have confined their researches to the space of dimension three. Concerning similar problems in spaces of higher dimension, Th. Reye declared:

"Die Geometrie van *n* Dimensionen verdient meines Erachtens diesen Namen nicht, sobald sie sich auf einen sogenannten Raum von mehr als drei Dimensionen bezieht; sie entbehrt aller Vorzüge der Anschaulichkeit und sinnlichen Darstellung, welche die eigentliche Geometrie auszeichnen."

However, in recognizing this view, Th. Reye has not been able to clarify completely the field defined by himself.

At the same time that F. Schur wrote the above-named paper, G Veronese (1) published an important memoir about the more-dimensional projective geometry, entitled: "Behandlung der projectivischen Verhältnisse der Räume von verschiedener Dimensionen durch das Princip des Projicirens und Schneidens".

In Chapter V of this memoir, "Erzeugnisse durch collineare Grundgebilde", G. Veronese formulates the general problem concerning a linear system of projectively related stars of primes in a space of arbitrary dimension.

The great importance of these investigations in connection with the problems in the space of three dimensions, may be seen from the account preceding G. Veronese's memoir, in which he writes: "Um eine Configuration von $n+1$ Punkten, oder eine Curve, oder eine 2-dimensionale Fläche, die gewisse Singularitäten besitzt, im gewöhnlichen Raume R_3 zu studieren, ist es in vielen Fällen nützlich, zuvörderst eine Configuration oder ein Gebilde 1^{ter} oder 2^{ter} Dimension in dem n-dimensionalen Raume R_n zu suchen, aus welchem mittelst geeigneten Projicirens oder Schneidens das gegebene Gebilde in eindeutiger Weise entsteht. Und zugleich kann man nicht nur jene Configuration, Curve oder Fläche, sondern auch eine Classe dieser Gebilde studiren, welche sämmtlich mittelst des Projicirens oder Schneidens aus jenem Gebilde im Raume R_n hervorgehen. Dieses Gebilde in R_n ist immer einfacher als das gegebene in R_3 und lässt sich daher besser behandeln".

The present investigation concerning the geometry of aggregates has been brought about thanks to the suggestions made by my tutor J. C. H. Gerretsen. He drew my attention to his general theory about linear sets of aggregates in a space of arbitrary dimension (J. C. H. Gerretsen (1)), containing the investigation begun by F. Schur and Th. Reye, and using the mathematical apparatus constructed by C. Segre in his paper "Sulle varietà che rappresentano le coppie di due piani o spazi" (C. Segre (1)). The notation and terminology which are used in this dissertation were also proposed by J. C. H. Gerretsen.

T. G. Room's (1) book, "The Geometry of Determinantal Loci" has also been very useful to this present investigation. Chapters III and IV of this book in particular have influenced the direction of this investigation. However, instead of the purely analytic method of T. Room, I have chosen a simpler miscellaneous synthetic-analytic method such as that used by G. Veronese and C. Segre in their previously mentioned papers. In this respect my exposition differs from that of Gerretsen, who has based the theory on the properties of correlations.

A summary of this dissertation now follows.

Some fundamental ideas in more-dimensional projective geometry are formulated in a preliminary chapter.

After a discussion about the Segre manifold, which is the representation of the two-fold projective space, a generalisation of this is given in Chapter I. This generalisation which is the so-called covariantly generated point-manifold $[m; n]_h$ in a space S_R, where $R = (m+1)(n+1) - 1$, is defined both analytically and synthetically. If a general point (z) of S_R has the homogeneous coordinate-vector

$$(z_{00}, \ldots, z_{0n}, z_{10}, \ldots, z_{1n}, \ldots, z_{m0}, \ldots, z_{mn}),$$

then $[m; n]_h$ consists of such points, which satisfy the condition that the $(m+1) \times (n+1)$-matrix $[z_{ij}]$ is of rank $\leqq h+1$.

This manifold can be generated synthetically by means of $n+1$ projectively related subspaces S_m of S_R, which are in mutually independent position.[1]

If we take the joins of corresponding subspaces S_h of the S_m, then we get a so-called covariant series $[m|n]_h$ of linear spaces. The set of points generated in this way, will be a manifold $[m; n]_h$. As $[m; n]_h$ coincides with $[n;m]_h$ a second covariant series $[n|m]_h$ can be distinguished on this manifold.

If we denote a general prime (ζ) of S_R by the coordinate-vector,

$$(\zeta^{00}, \ldots, \zeta^{0n}, \zeta^{10}, \ldots, \zeta^{1n}, \ldots, \zeta^{m0}, \ldots, \zeta^{mn}),$$

then the primes which satisfy the condition that the $(m+1) \times (n \times 1)$-matrix is of rank $h+1$, will form the so-called contravariantly generated prime-manifold $]m; n[_h$.

This manifold is covered by the two contravariant series $]m|n[_h$ and $]n|m[_h$, consisting of stars of primes $\Sigma_{(n+1)(h+1)-1}$ and $\Sigma_{(m+1)(h+1)-1}$ respectively. We denote the manifold of points which is formed by the vertices of the stars $\Sigma_{(n+1)(h+1)-1}$ of the series $]m|n[_h$, by $\overline{]m|n[_h}$, which we shall call a contravariantly generated manifold. It is proved that the contravariantly generated manifold $\overline{]m|n[_h}$ can at the same time be written as a covariantly generated manifold $[m; n]_{m-h-1}$.

Because a covariant correlation

$$p_{\mu\nu}\, \xi^\mu \eta^\nu = 0 \qquad (\mu = 0, \ldots, m;\ \nu = 0, \ldots, n)\ [2]$$

between two spaces S_m and S_n, where (ξ) and (η) represent primes in S_m and S_n respectively, is fixed by the $(m+1) \times (n+1)$-matrix $[p_{ij}]$, such a correlation can be represented by a point of S_R.

The nature of the correlation, which is determined by the rank

[1] For the definition of the conception, "in mutually independent position" see the Preparation.

[2] In this notation we use the summation convention, which says that whenever an index appears twice in an expression, once as a subscript and once as a superscript, it is to be summed over the entire range of that index.

of the matrix $[p_{ij}]$ appears again in the position of the image-point in S_R in relation to the manifolds $[m; n]_h$, where $h = 0, \ldots, min(m; n)$.

Likewise a contravariant correlation

$$\pi^{\mu\nu} x_\mu y_\nu = 0$$

between S_m and S_n, where (x) and (y) are points of S_m and S_n respectively, is determined completely by the position of its image-prime in S_R in relation to the manifolds $]m; n[_h$, where $h = 0, \ldots, \min(m; n)$.

If both matrices $[p_{ij}]$ and $[\pi^{ij}]$ satisfy the identity

$$p_{\mu\nu} \pi^{\mu\nu} = 0,$$

then the corresponding covariant and contravariant correlation are called apolar, which means that their image-point and image-prime in S_R are incident.

In chapter II an investigation is attempted of manifolds which appear as an intersection of the manifolds introduced in Chapter I, by a linear space.

If $\overline{]m|n[}_h$ is intersected by a space S_r, we denote this intersection by

$$\overline{]m|n[}_h \cap S_r. \quad ^1$$

Particular attention is drawn to the so-called pairing-theorems. Namely a subspace S_r of S_R can be situated in relation to the prime-manifold $]m; n[_h$ in such a way that S_r belongs to one or more primes of $]m; n[_h$.

A prime of $]m; n[_h$ belongs both to a star of the series $]m|n[_h$ and to a star of the series $]n|m[_h$.

Then the vertices of these stars have a special situation in relation to the space S_r, resulting in singularities on the manifolds $\overline{]m|n[}_h \cap S_r$ and $\overline{]n|m[}_h \cap S_r$ which are connected with $]m; n[_h \cap S_r$. Then a linear star of primes $]m[$ in S_r is defined by means of a contravariant correlation

$$\beta^{\mu\varrho} x_\mu z'_\varrho = 0$$

between a parametric space S_m and a fundamental space S_r. The star of primes gets a parametric representation and thus it represents a special entity in S_r. Such a configuration is called a *contravariant aggregate.*

If the matrix $[\beta^{\mu\varrho}]$ is of rank $m + 1$, then the contravariant aggregate $]m[$ will have both the virtual and effective extension m.[2]

[1] In Room (1) this manifold is denoted by $(|m + 1, n + 1|^{|}_{m-h}, [r])$.

[2] To the definition of these conceptions see the Preparation.

If, however, that rank is $m - i + 1$, then $]m[$ will have an effective extension $m - i$ and the aggregate is said to be i-ply degenerated. Likewise we can also define covariant aggregates $[m]$ in S_r. The trilinear relation

$$\omega^{\mu\nu\varrho} x_\mu y_\nu z_\varrho = 0 \qquad (\mu = 0, \ldots, m;\ \nu = 0, \ldots, n;\ \varrho = 0, \ldots, r)$$

defines a linear system of ∞^n contravariant aggregates $]m[$ in S_r. These form a series $]n|m[_0 \cap S_r$.

Since, by means of the connected matrices, this system of aggregates can be represented by a star Σ_m of primes in a space S_N, where $N = (m + 1)(r + 1) - 1$, an investigation of the degenerate aggregates in this system $]n|m[_0 \cap S_r$ will be an investigation of the position of Σ_n in S_N in relation to the manifolds $]m; r[_h$, where $h = 0, \ldots, \min(m; r)$.

In the same way we can represent a linear system of ∞^n covariant m-dimensional aggregates in S_r by a linear space S_n in S_N. Then the nature of this system is determined by the situation of S_n in relation to the manifolds $[m; r]_h$ in S_N, where $h = 0, \ldots, \min(m; r)$.

In Chapter III the trilinear relation

$$\omega^{\mu\nu\varrho} x_\mu y_\nu z_\varrho = 0$$

is considered in detail.

It can be interpreted both as an $]n|m[_0 \cap S_r$ and as an $]n|r[_0 \cap S_m$. From this the existence of a birational relation follows, between the manifolds $\overline{]m|n[_0} \cap S_r$ and $\overline{]r|n[_0} \cap S_m$.

Both systems of contravariant aggregates can be represented by a star Σ_n in the matrix-space S_N, where $N = (m + 1)(r + 1) - 1$. This image-star has as a vertex an S_{N-n-1}, which with the manifold $[m; r]_0$ gives an intersection $[m; r]_0 \cap S_{N-n-1}$. This intersection appears to be birationally related to both the manifolds $\overline{]m|n[_0} \cap S_r$ and $\overline{]r|n[_0} \cap S_m$.

Since S_{N-n-1} in S_N also represents a linear system of ∞^{N-n-1} m-dimensional covariant aggregates in S_r, $[m; r]_0 \cap S_{N-n-1}$ represents such samples of this system which are of maximum degeneration. By dualizing we also get the degenerated samples in a system $]N\text{-}n\text{-}1|m[_0 \cap S_r$.

The systems $]n|m[_0 \cap S_r$ and $]N\text{-}n\text{-}1|m[_0 \cap S_r$, which are connected with each other in the above-mentioned way are called "complementary".

In his series of papers, Th. Reye has instituted the first investigation into linear systems of one-, two- and three-dimensional systems of contravariant aggregates in S_3.

In his book "Punktreihengeometrie" E. Weiss (1) has considered the linear system of one-dimensional contravariant aggregates in S_3 in detail, which he calls "Ebenenreihen". By means of the theory of invariants, composed by E. Study, E. Weiss has penetrated deeply into this domain. However, this apparatus seems to fail for systems of aggregates of higher dimension.

The following investigation begins with a test by the general theory developed in the first three chapters, of the results which Th. Reye obtained from systems of two-dimensional contravariant aggregates in S_3.

Chapter IV and V consider consecutively the systems $]1|2[_0 \cap S_3$ and $]2|2[_0 \cap S_3$.

In Chapter VI a short summary is given of the results obtained from the systems $]3|2[_0 \cap S_3$ and $]4|2[_0 \cap S_3$.

PREPARATION

As an introduction into the geometry of aggregates, a brief survey of some fundamental ideas in projective geometry will follow here. For the proofs, which are extremely limited here, we refer to the literature in this field.

§ 1. The projective r-dimensional space S_r.

Starting with $r + 1$ points $A^0, \ldots, A^r$ given in a projective space, an arbitrary element of the subspace, spanned by these $r + 1$ points, can be represented by

$$x_0 A^0 + \ldots + x_r A^r = x_\lambda A^\lambda \qquad (1.1)^1$$

The numbers $x_0, \ldots, x_r$ are taken from the field of complex numbers and do not vanish simultaneously. If all the numbers x_ϱ are equal to zero, then the space represented by (1.1) is called empty. We denote the space spanned in this way by $[r]$, calling r the *virtual dimension* of this space. Then the empty space will be of dimension –1.

If the $r + 1$ points $A^0, \ldots, A^r$ are linearly independent, then the effective dimension of the space is r.

However, if i points depend on the $r + 1 - i$ remaining points which are linearly independent, the effective dimension of the space is $r - i$ and we call i the order of degeneration of $[r]$. If $[r]$ is not degenerated, then it will also be denoted by S_r.

Of the point $x_\lambda A^\lambda$, where $A^0, \ldots, A^r$ span the space S_r we call $x_0, \ldots, x_r$ coordinates of the point. These numbers do not vanish simultaneously and are determinate up to a multiplicative non-zero factor, provided a point has been assigned which has the coordinates $(1, \ldots, 1)$. Now the points A^λ are called the angular points of the simplex of reference of the space which together with a unit point U is the foundation of the coordinate system. The coordinates of these points are:

[1] Here the summation convention is used, as defined in the Introduction.

$$\begin{aligned} A^0 &= (1, 0, 0, \ldots, 0) \\ A^1 &= (0, 1, 0, \ldots, 0) \\ &\text{-------------} \\ A^r &= (0, 0, 0, \ldots, 1) \\ U &= (1, 1, 1, \ldots, 1) \end{aligned}$$

The representation, as stated above, is not unique, as we are free to select the simplex of reference. The coordinate-vectors $(x_0, \ldots, x_r)$ and $(y_0, \ldots, y_r)$ of the same point are always related by a linear transformation

$$y_\varkappa = a^\nu_\varkappa x_\nu \quad (\varkappa, \nu = 0, \ldots, r)$$

with determinant $|a^\nu_\varkappa| \neq 0$.

§ 2. The dual space.

We consider in the space S_n those points of which the coordinates $x_0, \ldots, x_n$ satisfy a linear relation

$$\xi^\nu x_\nu = 0, \quad (\nu = 0, \ldots, n) \tag{2.1}$$

The numbers ξ^ν are not all zero.

Such a set of points is called a prime of S_n. Supposing $\xi^i \neq 0$ applies to the prime (2.1), then for every point of this prime we can write x_i as a linear combination of

$$x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n.$$

We get all the points of this prime by the variables $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ running over the whole field of the complex numbers, in such a way that they do not vanish simultaneously; so x_i is determinate at every choice. So the prime, considered as a space of points, is an $n-1$ -dimensional subspace of S_n.

A prime of S_n is determined by the homogeneous coordinate-vector

$$(\xi^0, \ldots, \xi^n).$$

If we consider the primes as primitive elements, the set of all primes of S_n can be interpreted as a projective space. This space is called the dual one of S_n. Hence the primes of S_n are the points of this dual space.

If we now select primes

$$A_0, \ldots, A_r$$

from S_n, we can represent an arbitrary element of the subspace, spanned by these $r + 1$ elements, by

$$\xi^0 A_0 + \ldots + \xi^r A_r = \xi^\lambda A_\lambda \tag{2.2}$$

The remarks of § 1 about subspaces are also valid in the dual space, with the difference that subscripts are transformed into superscripts and conversely. For the sake of clearness the terminology is changed: A subspace will be a star; instead of dimension we speak about extension.

If the base $A_0, \ldots, A_r$ of this star is linearly independent, then the star has an effective extension r and we denote $]r[$ by Σ_r also. Then $\xi^0, \ldots, \xi^r$ represent the homogeneous coordinates again. It is possible to select the fundamental system of the dual space arbitrarily. However, the relation of incidence

$$\xi^\varrho x_\varrho = 0 \quad (\varrho = 0, \ldots, r) \tag{2.3}$$

which gives a relation between a point of S_r and a point of Σ_r, which is the dual of S_r, is invariant only if the fundamental elements of Σ_r are the subspaces of the simplex of reference in S_r, opposite to the angular points, and the unit element Ω of Σ_r is the harmonic prime of the unit point U of S_r in relation to the fundamental simplex.

The base elements of Σ_r are:

$$\begin{aligned} A_0 &= (1, 0, 0, \ldots, 0) \\ A_1 &= (0, 1, 0, \ldots, 0) \\ &- - - - - - - - - - - - \\ A_r &= (0, 0, 0, \ldots, 1); \end{aligned}$$

the unit element of Σ_r is $\Omega = (1, 1, 1, \ldots, 1)$.

Just as in § 1 the representation (2.2) is not unique, but depends on the selection of the base.

However, it is determined within a linear transformation of which the matrix is of maximal rank.

§ 3. Subspaces of S_r; substars of Σ_r.

Considering $k + 1$ linearly independent points of S_r, $(0 \leqq k < r)$, all the points which we can represent by a linear combination of these $k + 1$ points, span a space S_k which we call a subspace of S_r.

Since the points of the S_k are determinate by $k + 1$ homogeneous coordinates, S_k can also be obtained by considering such points of S_r, which satisfy $r - k$ linear conditions

$$\xi_\lambda^\nu x_\nu = 0, \quad (\nu = 0, \ldots, r; \lambda = 1, \ldots, r - k) \tag{3.1}$$

with a matrix $[\xi_\lambda^\nu]$ of maximal rank $r - k$, which means that the conditions are linearly independent.

Every individual condition determines a prime of S_r. Thus the conditions together yield a Σ_{r-k-1}.

The points of S_k belong to all the elements of Σ_{r-k-1}, so that the star Σ_{r-k-1} has a kernel space S_k.

Considering two subspaces S_h and S_k of S_r, the set of all the points belonging both to S_h as well as to S_k, is called the intersection of S_h and S_k; the join of these two spaces is the linear subspace of S_r of minimal dimension containing both S_h as well as S_k. Concerning the intersection and join, the following rules are valid.[1]

1° The intersection of two linear subspaces S_h and S_k of S_r is either empty or again a subspace of S_r.

2° If the intersection of S_h and S_k is empty, then their join is an S_{h+k+1}. If S_h and S_k intersect each other in an S_p, then their join is an S_{h+k-p}.

If the intersection of S_h and S_k is empty, we say S_h and S_k are in mutually independent position.

S_h and S_k are in general position to each other if they are either in mutually independent position or have as a join the entire space S_r.

If S_h and S_k in S_r are in general position and $h + k \geqslant r$, then the intersection of S_h and S_k is a space S_{h+k-r}.

The conceptions of intersection and join may be defined in the same way in the case of more than two linear subspaces of S_r. The subspaces $S_{h_1}, \ldots, S_{h_n}$ of S_r are said to be in a mutually independent position, if every one of these subspaces is in a mutually independent position with the join of all the other subspaces.

An element S_{h_i} from the set of subspaces $S_{h_1}, \ldots, S_{h_n}$ of S_r is said to be in general position with regard to the other elements of the set, if S_{h_i} is in general position with regard to the join of every arbitrarily chosen subset of spaces, taken from the set $S_{h_1}, \ldots, S_{h_{i-1}}, S_{h_{i+1}}, \ldots, S_{h_n}$. The subspaces $S_{h_1}, \ldots, S_{h_n}$ of S_r are in mutually general position, if every one of this set is in general position with regard to the other elements of the set.

Dualizing we get:

1° $k + 1$ linearly independent elements of a star Σ_r span a Σ_k, so that this star has a kernel space S_{r-k-1}.

[1] For proofs see Bertini (1) Chap. I.

2° The intersection of two linear substars Σ_h and Σ_k of Σ_r is either empty or a linear substar of Σ_r.

3° If the intersection of Σ_h and Σ_k is empty, their join is a Σ_{h+k+1}; if their intersection is a Σ_p, then they have a join Σ_{h+k-p}.

We can define the conception "mutually independent position" and "mutually general position" of substars of Σ_r in the same way as we have done for subspaces of S_r.

§ 4. Algebraic manifolds.

We will use here the conception "algebraic manifold" in an intuitive way such as occurs for instance in the text-book of J. G. Semple and L. Roth. (1)

For an exact treatment refer to W. Hodge and D. Pedoe (1), Chap. X.

4.1. Primals. We shall give the definition of a primal as a generalisation of that of a prime in a projective space S_r. The locus of points in S_r, whose coordinates satisfy the relation

$$F(x_0, \ldots, x_r) = 0 \tag{4.1.1}$$

whereby F is a homogeneous polynomial with coefficients in the field of the complex numbers, is called a primal. Because these points satisfy one relation the manifold is said to have the dimension $r-1$. If F is of degree n we agree that the primal is of order n and is denoted by the symbol V^n_{r-1}. If V^n_{r-1} is intersected by a general line, determined by the two points $P = (p_0, \ldots, p_r)$ and $Q = (q_0, \ldots, q_r)$, the system of equations

$$\begin{cases} F(x_0, \ldots, x_r) = 0 \\ x_\varkappa = \lambda p_\varkappa + \mu q_\varkappa, \quad (\varkappa = 0, \ldots, r) \end{cases} \tag{4.1.2}$$

have n solutions in the ratio $\lambda : \mu$, so that a line meets V^n_{r-1} in n points which do not necessarily differ.

So we can also define the order of a primal as the number of points of intersection of the primal and a general line, whereby every point of intersection is provided with a suitable multiplicity. If a point of the manifold V^n_{r-1} is such that a general line through that point meets V^n_{r-1} in $n-k$ further points, then that point is said to be a multiple point with multiplicity k.

In the event that the polynomial $F(x_0, \ldots, x_r)$ is reducible the primal is also reducible and consists of the primals which are obtained by separately making every irreducible factor of F equal to zero.

If F is irreducible, the primal is also called irreducible. Considering r general primals $V^{n_\lambda}_{r-1}$ in S_r ($\lambda = 1, \ldots, r$), the points belonging to all these primals satisfy the simultaneous equations

$$F_\lambda (x_0, \ldots, x_r) = 0 \qquad (\lambda = 1, \ldots, r) \tag{4.1.3}$$

which represent the primals respectively.

According to the theorem of Bezout [1] these r equations have in general a finite number of solutions $\prod_{\lambda=1}^{r} n_\lambda$, so that the r primals have $\prod_{\lambda=1}^{r} n_\lambda$ common points.

The dual results of those stated above, are obvious.

4.2. Manifolds of dimension less than $r - 1$. We call the set of points in an S_r an algebraic manifold if the coordinates of those points satisfy a number of equations

$$F_\lambda (x_0, \ldots, x_r) = 0 \tag{4.2.1}$$

whereby F_λ are homogeneous polynomials with coefficients in the field of the complex numbers.

We can interpret this manifold to be the intersection of primals in S_r, generated by every one of the individual equations $F_\lambda = 0$. We can define the dimension of an algebraic manifold in such a way that if the manifold has only a finite number of points in common with a general S_k, then the manifold is of dimension $d = r - k$.

If the manifold V, defined as stated above, can be considered point-set theoretic as the sum of two algebraic manifolds W and U, both different from V, then V is called reducible. In the opposite case V is said to be irreducible. If V is reducible, then the manifold consists of a number of irreducible components.

Among these irreducible components manifolds of dimension less than $r - k$ may appear, but at least one of the components is of dimension $r - k$.

A frequently used criterion of the irreducibility of a manifold V over a ground field K is that V contains a generic point.[2] Such a generic point is defined as follows:

[1] See Semple and Roth (1): Introduction.

[2] See Hodge and Pedoe (1): Chap. X.