LECTURES ON THE THEORY OF FUNCTIONS

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NOTE

The theorems of the Introduction have numbers below 100, those of Chapter I begin at 101, those of Chapter II at 201. Sections are numbered consecutively.

PREFACE

The Introduction and Chapter I were printed off in 1931 and some important changes should be supplied from the Addenda and Corrigenda, which an intending reader should take note of at once. Chapter II, whose completion has been unavoidably delayed, has now been rewritten. For help in this I owe an overwhelming debt to Dr. W. W. Rogosinski, who not only supplied much of the material, but criticised and corrected my text in the last detail.

I wish also to express my gratitude to the printers Messrs. C. F. Hodgson & Son for their courtesy and great forbearance over a difficult 20 years.

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CONTENTS

Introduction.

THE various matters collected in the Introduction agree only in being more conveniently separated from their applications. It is not, however, necessary to read it consecutively, and much of it is first required in Volume 2; the reader may therefore welcome a few words of explanation and advice.

He cannot become familiar too early with the inequalities of Hölder and Minkowski, and he should read consecutively (but not try to memorize) to the end of Section 2 if he can do so without becoming impatient. This section is developed rather more systematically than is necessary for applications, but the number of distinct forms in Theorems 1 and 2 that are specifically used is surprisingly large; and if the details are taken with a judicious lightness the subject is quite an easy one. Section 3 is very short. There is a certain field of complex function theory (the problems of "boundary-values"—these are discussed in Volume 2) which demands a fairly complete "real-variable" technique. Sections 4 and 5 are designed to meet this need, Section 4 dealing with general theory, and Section 5 with the more special subject of Fourier series. While not exhaustive, the account is sufficiently systematic to be read for its own sake, but the reader may postpone it if he wishes until he reaches Volume 2 Section 6 is concerned with an isolated problem of *analysis situs*, and may be read when it becomes relevant (in Section 19). Section 7 presents a fairly complete general theory of harmonic functions; much of it is required later, it is easy, and the subject is apt to be neglected in England; it should probably be read before Chapter I. Section 8 consists of straightforward calculations. It sets out the behaviour of certain special functions whose rôle is to be illustrative, and especially to provide "Gegenbeispiel" 's†. It is required hardly at all in Volume 1.

† A "Gegenbeispiel" for a proposition p is an example which shows that p is false: the function x^{-1} is a "Gegenbeispiel" for the proposition "all functions are bounded in $0<x<1$". The important examples are those which complete the account of a theorem by showing that it is "best possible" (depends on the minimum hypotheses).

1. *Notation*. We use the symbol $A(x, y, \ldots)$, or sometimes $A_{x, y, \ldots}$, for a positive constant depending only on the parameters shown explicitly; in particular A will denote a positive absolute constant. We use K for a positive constant depending in general on all the porameters of the context. We use ϑ for a number satisfying $|\vartheta| \leqslant 1$. A's, K's, and ϑ's are not in general the same from one occurrence to another; if we wish to preserve their identity in the course of an argument we affect them with suffixes 1, 2, ...

$\epsilon(x)$, ϵ_n, etc., denote functions tending to 0 as their argument tends to the limit (finite or infinite) under consideration. The symbol $o(1)$ is available for such functions, and the ϵ notation is used only to mark a distinction; we use it for functions that are independent of some parameter or parameters.

The symbol ϵ without an argument, and also δ, denote as usual positive constants ("arbitrarily small").

Certain letters *used as indices* (exponents) will denote numbers subject to special conditions. μ may be any real constant, positive or negative. The remaining letters denote *positive* constants, and, moreover, are restricted by the following inequalities:

$$\lambda > 0,\ k \geqslant 1,\ r > 1;\quad 1 < p \leqslant 2,\ q \geqslant 2;\quad 0 < \kappa \leqslant 1,\ 0 < \rho < 1.$$

We shall occasionally allow ourselves the licence of extending the ranges of λ, κ, ρ to include 0, those of μ, λ, k, r, q to include $+\infty$ and that of μ to include $-\infty$; but in such cases we shall always indicate the extension explicitly. [The commonest indices are λ and r. p and q do not occur in Vol. 1. The definitions are repeated from time to time, and the reader need not memorize them.]

We write $t' = t/(t-1)$, where t is any one of the special indices (supposed, however, not to have the value $t = 1$). A dashed letter does not necessarily belong to the class denoted by the undashed letter: thus p' and q' are respectively of types q and p, and λ', κ', ρ' may be negative.

The relation between t and t' may be expressed in two further ways, with which the reader should make himself familiar

$$\frac{1}{t} + \frac{1}{t'} = 1, \quad (t-1)(t'-1) = 1.$$

The integrals with which we shall be concerned are generally extended over a bounded set of points. Such a set of points can be reduced by a trivial transformation to lie within any given interval: we shall suppose always, unless the contrary is stated (and this does sometimes happen), that all sets E, e, ... are contained in the interval $(-\pi, \pi)$, which we denote by E_0. We write $E_1 \subset E_2$ for "E_1 is contained in E_2", and denote $E_0 - E$ by CE.

By HK, the "product" of two sets of points H and K, we mean the set of points common to H and K.

A function $f(\theta)$ to be considered in E_0 is likely to have some natural relation to the period 2π. On balance it pays to lay down the convention that "f is continuous in E_0" shall include the relation $f(-\pi) = f(\pi)$. In theorems about functions not necessarily continuous it is generally possible to alter arbitrarily the value of the function at a single point. In such circumstances we shall tacitly suppose that $f(-\pi) = f(\pi)$ and that f, defined originally in E_0, exists everywhere and has the period 2π. This convention enables us, for example, to treat an interval $|\theta - \theta_0| \leqslant k$ on the same footing when it projects out of E_0 as when it does not.

Unless the contrary is stated all *given* functions are supposed measurable: other questions of measurability are generally trivial, and we do not discuss them.

When $|f(\theta)|^\lambda$ is integrable in the sense of Lebesgue in a set E we say that f belongs to the class L^λ in E. We write also for brevity L in place of L^1.

The "sign of z", or, in symbols, $\operatorname{sgn} z$, is defined to be 0 if $z = 0$ and $z/|z|$ otherwise. $\bar{z}$ denotes the conjugate of z, $\overline{\operatorname{sgn} z} = \operatorname{sgn} \bar{z}$.

The symbol $[f]_N$ denotes f if $|f| \leqslant N$, and $N \operatorname{sgn} f$ if $|f| > N$. $[E]_N$ denotes that part of the set E for which the modulus of the variable does not exceed N.

By a null-set we understand a set of zero measure, by a null-function a function that is zero except in a null-set. $f \equiv \phi$, or, in words, "f is equivalent to ϕ", means that $f = \phi$ except in a null-set, or that $f - \phi$ is a null-function.

We shall use the following abbreviations :—

p.p. ("presque partout") for "almost everywhere" or "almost always" (*i.e.* "except in a null-set"). ["a.e." is insufficiently vivid and is apt to be mistaken for other things];

b.v. for "bounded variation" and "of bounded variation";

a.c. for "absolute continuity" and "absolutely continuous";

u.b.v. and u.a.c. for "uniform(ly) b.v." and "uniform(ly) a.c.";

t.v. for "total variation".

By a "trigonometrical polynomial" we understand a finite sum of type

$$\sum_{n=0}^{N} (c_n \cos n\theta + d_n \sin n\theta).$$

2. *The inequalities of Hölder and Minkowski.*

2 . 1. We suppose until further notice, unless the contrary is stated, that all letters denote numbers that are positive or zero. The sums with which we deal are in general taken over an infinity of terms, but in our *proofs* we may suppose them finite, and complete the argument by a trivial passage to the limit. There is a single exception to this rule: Theorem 4 of § 2 . 82. Here "convergence" is mentioned explicitly and given a special treatment.

Hölder's inequality is

$$(H) \qquad \Sigma ab \leqslant (\Sigma a^r)^{1/r} (\Sigma b^{r'})^{1/r'} \quad (r > 1).$$

Minkowski's inequality is

$$(M) \qquad \left(\Sigma (a+b)^k\right)^{1/k} \leqslant (\Sigma a^k)^{1/k} + (\Sigma b^k)^{1/k} \quad (k \geqslant 1).$$

We first prove these results, then develop them at length, and finally collect everything for reference in Theorems 1 and 2.

2 . 2. Let $U^r = \Sigma a^r$, $V^{r} = \Sigma b^{r'}$, $W = \Sigma ab$. We have

$$(1) \qquad ab \leqslant \frac{a^r}{r} + \frac{b^{r'}}{r'}.$$

For

$$\left(\frac{a^r}{r'} + \frac{b^{r'}}{r}\right) \Big/ ab = t(x) = \frac{x^r}{r} + \frac{x^{-r'}}{r'},$$

where

$$x = a^{1/r'} b^{-1/r},$$

and differentiation shows that $t(x)$ is a minimum (for $x \geqslant 0$) when $x = 1$, in which case $t = 1$.

It follows from (1) that if λ is any positive constant

$$ab = \lambda a . \lambda^{-1} b \leqslant \lambda^r \frac{a^r}{r} + \lambda^{-r'} \frac{b^{r'}}{r'}.$$

Summing we have

$$(2) \qquad W \leqslant \lambda^r \frac{U^r}{r} + \lambda^{-r'} \frac{V^r}{r'}.$$

We may suppose in (H) that $U, V > 0$, in which case, if we choose λ so that

$$\lambda^r U^r = \lambda^{-r'} V^{r'} = (\lambda^r U^r)^{1/r} (\lambda^{-r'} V^{r'})^{1/r'} = UV,$$

(2) becomes

$$W \leqslant \frac{UV}{r} + \frac{UV}{r'} = UV,$$

and this is (H).

The inequality (M) is trivial when $k=1$; supposing then $k>1$ we have, by (H),

$$\begin{aligned} T^k &= \Sigma(a+b)^k = \Sigma(a+b)^{k-1}a + \Sigma(a+b)^{k-1}b \\ &\leqslant \{\Sigma(a+b)^k\}^{1/k'}(\Sigma a^k)^{1/k} + \{\Sigma(a+b)^k\}^{1/k'}(\Sigma b^k)^{1/k} \qquad (3) \\ &= T^{k-1}\{(\Sigma a^k)^{1/k} + (\Sigma b^k)^{1/k}\}, \end{aligned}$$

and the desired result follows.

(M) evidently extends directly (or by induction) to more than two sets of numbers (a), (b); we have, in fact,

$$\{\Sigma(a+b+c+\ldots)^k\}^{1/k} \leqslant (\Sigma a^k)^{1/k} + (\Sigma b^k)^{1/k} + \ldots . \qquad (4)$$

Results corresponding to (H) and (M) exist also with integrals in place of sums, and in (4), where a double summation is involved, there are also mixed forms. For the most part the proofs are substantially the same for sums or integrals; where this is so we shall generally give only the *argument* for sums; where it is not the integral case is the more difficult and we consequently select it. In *stating* results we select sometimes the sum, sometimes the integral form. We suppose in our proofs that the range of integration is bounded; extensions to infinite range are trivial when they are valid, and we do not consider them until our final summing up. Our integrals are Lebesgue integrals. We actually require none but elementary integrals in Volume 1, but the subject is more *easily* treated in the general field, and the full results are, in any case, required in Volume 2.

For the "integral-integral" form of (4) the argument transforms as follows: The case $k=1$ is trivial. Supposing then $k>1$ we have

$$T^k = \int dy \left(\int f(x,y)\,dx\right)^k = \int de \left\{\int f \left(\int f dx\right)^{k-1} dy\right|$$

$$\leqslant \int dx \left\{\left[\int f^k dy\right]^{1/k} \left[\int \left(\int f dx\right)^k dy\right]^{1/k}\right\} = \int dx \left[\int f^k dy\right]^{1/k} T^{k/k'}, \qquad (5)$$

or

$$T \leqslant \int dx \left(\int f^k dy\right)^{1/k}$$

which is the desired result.

2.3. Let now f and g be functions, possibly complex, for which $g \not\equiv 0$. Then

$$\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} fg\,d\theta\right| \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^r d\theta\right)^{1/r}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|g|^{r'} d\theta\right)^{1/r'} = M_r(f)\,M_{r'}(g). \qquad (1)$$

This is, in fact, what may be called the "mean" form of (H) (for integrals). The 2π's may be retained or omitted at our pleasure, since they occur to the same power -1 on both sides. We prove now that the sign of equality in (1) holds† if and only if each of

$$(2)\qquad |f|^r \equiv c|g|^{r'}, \quad \text{where} \quad c = M_r^r(f)/M_{r'}^{r'}(g),$$

and

$$(3)\qquad \operatorname{sgn} fg \equiv e^{i\alpha} \equiv c\,|\,g\,|^{r'}, \quad \text{where } \alpha \text{ is a real constant,}$$

hold in the set of θ for which $f \neq 0$.

It is easily seen that equality does hold in (1) subject to (2) and (3). Suppose now that equality holds. Then, in the first place, it continues to hold when the integrand fg is replaced by $|f|\,|g|$. Let $\lambda^{r+r'} = c^{-1}$. Then (indeed for any λ)

$$(4)\qquad |fg| \leqslant \lambda^r \frac{|f|^r}{r} + \lambda^{-r'} \frac{|g|^{r'}}{r'},$$

and equality in (4) happens only if $|f|^r = c|g|^{r'}$. If (2) is false there exists a set, not null, in which (4) holds with inequality, and therefore a non-null set in which the difference of the two sides exceeds some positive δ‡. Then

$$\int_e |fg|\,d\theta < \frac{\lambda^r}{r}\int_e |f|^r\,d\theta + \frac{\lambda^{-r'}}{r'}\int_e |g|^{r'}\,d\theta.$$

Since in any case

$$\int_{E_0-e} |fg|\,d\theta \leqslant \frac{\lambda^r}{r}\int_{E_0-e} |f|^r\,d\theta + \frac{\lambda^{-r'}}{r'}\int_{E_0-e} |g|^{r'}\,d\theta,$$

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† The reader will find in Vol. 2 that the conditions for equality (here and in §2.76) can be important weapons of argument: it is a mistake to suppose that they are of purely academic interest.

‡ We shall often have to use the principle involved here, which is that if $\phi(\theta) > 0$ in a set of positive measure, then, for some δ, $\phi > \delta$ in a set of positive measure. The principle can be generalized into the following form.

Suppose that with every θ of a set E of positive measure there are associated h positive numbers $\phi_1(\theta), \phi_2(\theta), \ldots, \phi_h(\theta)$; k finite real numbers $M_1(\theta), M_2(\theta), \ldots, M_k(\theta)$; and l positive integers $N_1(\theta), N_2(\theta), \ldots, N_l(\theta)$. Then there exists a positive number a, a finite μ, l positive integers $\nu_1, \nu_2, \ldots, \nu_l$, all independent of θ, and a perfect set E^* of positive measure contained in E, such that for every θ of E^*

$$\phi_n(\theta) > a \;\; (n \leqslant h), \quad |M_n(\theta)| < \mu \;\; (n \leqslant k), \quad N_n = \nu_n \;\; (n \leqslant l).$$

In fact, let $H(p, q : r_1, r_2, \ldots, r_l)$ be the set of θ of E for which $\phi_n(\theta) > p^{-1}$ $(n \leqslant h)$, $|M_n(\theta)| < q$ $(n \leqslant k)$, $N_n = \nu_n$ $(n \leqslant l)$. Every θ of E belongs to some set H, and $E = \Sigma H$, the summation being taken over all positive integral $p, q, r_1, \ldots$. Since Σ has a denumerable number of terms, some H has positive measure with E, since $mE \leqslant \Sigma mH$. H contains a perfect set E^* of positive measure, and this satisfies the required conditions, with

$$a = p^{-1}, \quad \mu = q, \quad \nu_n = r_n.$$

we have by addition

$$\frac{1}{2\pi}\int_{E_0}|fg|\,d\theta < \frac{\lambda^r}{r}\frac{1}{2\pi}\int_{E_0}|f|^r d\theta + \frac{\lambda^{-r'}}{r'}\frac{1}{2\pi}\int_{E_0}|g|^{r'} d\theta$$

$$= \left(\frac{1}{r}+\frac{1}{r'}\right) M_r(f)\,M_{r'}(g) = M_r(f)\,M_{r'}(g),$$

contrary to hypothesis. Thus (2) must hold.

Finally, for equality in (1) we must have

$$\int_{E_0} fg\,d\theta = e^{i\beta}\int_{E_0}|fg|\,d\theta$$

$$\int_{E_0}|fg|\,(1-e^{-i\beta}\operatorname{sgn} fg)\,d\theta = 0.$$

The real part of the integrand being non-negative, we must have

$$|fg|\,\{1-\Re(e^{-i\beta}\operatorname{sgn} fg)\} \equiv 0.$$

Since the set in which $fg = 0$ is equivalent, by (2), to the set in which $f = 0$, we have $\Re(e^{-i\beta}\operatorname{sgn} fg) \equiv 1$ and so $e^{-i\beta}\operatorname{sgn} fg \equiv 1$, except when $f = 0$. This completes the proof.

The case of sums is much simpler.

Consider now the case of equality in the (M) inequalities, supposing everything non-negative. There is equality in all cases if $k = 1$. If $k > 1$ the condition for equality in the "integral-integral" form is that $f(x, y) = F(x)\,G(y)$ p.p. in x and p.p. in y. In fact, for equality in (5) of § 2 . 2 the x-integrands must be equal p.p. in x. By the (H) result equality requires

$$\frac{\{f(x,y)\}^k}{\left\{\left(\int f\,dx\right)^{k-1}\right\}^{k'}} = c(x),$$

p.p. in y, where $c(x)$ is independent of y. This proves the result.

In the "sum-sum" form (4) the condition is that $b_n = c a_n$, $c_n = c' a_n$, ... for all n, where c, c', ... are positive constants.

2 . 4. The inequality (H) remains valid if the index r is replaced by a $\mu < 1$ and the sign of inequality is reversed, provided only that $\mu \neq 0$ [negative values of μ are permitted]. Similarly (M) is true if k is replaced by $\mu \leqslant 1$ and the sign of inequality is reversed, provided $\mu \neq 0$.

To prove this let us denote the inequality (H) by $I(a, b, r)$, and the inequality with reversed sign by I^*. If now $\mu = -\lambda < 0$ and we write $a = \alpha^{-(\lambda+1)/\lambda}$, $b = (\alpha\beta)^{(\lambda+1)/\lambda}$, $I^*(a, b, -\lambda)$ is equivalent to $I(\alpha, \beta, 1+\lambda^{-1})$, which is true If $\mu = \rho$ $[0 < \rho < 1]$ and we write $a = (\alpha\beta)^{1/\rho}$, $b = \beta^{-1/\rho}$, then $I^*(a, b, \rho)$ is equivalent to $I(\alpha, \beta, 1/\rho)$, which is true. Thus our assertion about (H) is proved. In the case of (M) we have only to carry out our former proofs, using I^* in place of I.

2.5. LEMMA α. $$(\Sigma a) \geqslant (\Sigma a^k)^{1/k} \quad (k \geqslant 1).$$

For $$(\Sigma a)^k = \Sigma\{(\Sigma a)^{k-1}.a\} \geqslant \Sigma\{a^{k-1}.a\}.$$

[The simplest case of the lemma is

$$(1+x)^k \geqslant 1+x^k \quad (x \geqslant 0).]$$

The inequality (H) extends at once to the form

(1) $$|\Sigma ab \ldots| \leqslant (\Sigma |a|^{r_1})^{1/r_1} (\Sigma |b|^{r_2})^{1/r_2} \ldots,$$

where the r's are connected by

(2) $$\Sigma \frac{1}{r} = 1,$$

and the a's, b's, ... are not necessarily positive. To prove this we write the product $ab \ldots$ as $\alpha\beta$ and use (H) with $r = r_1$. In the sum $\Sigma\beta^{r_1'}$ we now write β as $b\gamma$ and use (H) with $r = r_2$, and so on. We thus obtain (1).

We observe next that (1) remains true subject only to

(3) $$r_1 > 1, \; r_2 > 1, \; \ldots, \quad \Sigma \frac{1}{r} \geqslant 1.$$

In fact, let $\Sigma 1/r = k$, or $\Sigma 1(rk) = 1$. Then, by Lemma α,

$$|\Sigma ab \ldots t| \leqslant (\Sigma |a|^{1/k} \ldots |t|^{1/k})^k \leqslant [\Pi \{\Sigma(|a|^{1/k})^{rk}\}^{1/(rk)}]^k = \Pi(\Sigma |a|^r)^{1/r}.$$

The inequality (1), *subject to* (2), may be replaced by the "mean" form

$$\left|\frac{1}{n} \Sigma ab \ldots\right| \leqslant \Pi \left(\frac{1}{n} \Sigma |a|^r\right)^{1/r},$$

in which n is the number of terms in each set of numbers (a), (b), ... *This* inequality does not hold subject to (3). We shall call inequalities "homogeneous"† when they are true equally in "sum (integral)" or "mean" form.

We conclude this paragraph by noting some easy variants and consequences of (H). The integrals are all taken over $(-\pi, \pi)$, and f, g, ... are not necessarily positive.

$$\left|\frac{1}{2\pi}\int fg\,d\theta\right|^k \leqslant \left(\frac{1}{2\pi}\int |f^k g|\,d\theta\right)\left(\frac{1}{2\pi}\int |g|\,d\theta\right)^{k-1} \quad (k \geqslant 1).$$

† The homogeneity is in the range of summation or integration.

[Trivial for $k=1$, otherwise a consequence of (H) with f replaced by $|fg^{1/k}|$, g by $|g|^{1/k}$.]

(5) $$\left|\frac{1}{2\pi}\int fgh\,d\theta\right| \leqslant \left(\frac{1}{2\pi}\int|f^r h|\,d\theta\right)^{1/r}\left(\frac{1}{2\pi}\int|g^{r'}h|\,d\theta\right)^{1/r'} \quad (r>1).$$

(6) $$\left|\frac{1}{2\pi}\int fg\,d\theta\right| \leqslant \left(\frac{1}{2\pi}\int|f^h g^k|\,d\theta\right)^{1/r}\left(\frac{1}{2\pi}\int|f|^h d\theta\right)^{1/s}\left(\frac{1}{2\pi}\int|g|^k d\theta\right)^{1/t}$$

if $$h\geqslant 1,\quad k\geqslant 1,\quad h+k>hk,$$

where $$\frac{1}{r}=\frac{1}{h}+\frac{1}{k}-1,\quad \frac{1}{s}=1-\frac{1}{k},\quad \frac{1}{t}=1-\frac{1}{h}.$$

[For $|fg|=|f^h g^k|^{1/r}.(|f|^h)^{1/s}.(|g|^k)^{1/t}$].

(7) $$\left|\frac{1}{2\pi}\int f\,d\theta\right| \leqslant \left(\frac{1}{2\pi}\int|f|^k d\theta\right)^{1/k} \quad (k\geqslant 1).$$

This is the special case $g=1$ of (4). The parallel form is

(8) $$\left|\frac{1}{n}\Sigma a\right| \leqslant \left(\frac{1}{n}\Sigma|a|^k\right)^{1/k}.$$

(8) is not homogeneous [nor is (7)]. If we suppress the factors $1/n$ the inequality becomes false; indeed, when the a's are non-negative, it becomes true with sign reversed, as is seen at once from Lemma a. (The result corresponding to this in the theory of integrals has little interest.)

2.6. We prove next (a's and b's not necessarily positive):

(1) $$(\Sigma|a|^k)^{1/k}-(\Sigma|b|^k)^{1/k} \leqslant (\Sigma|a+b|^k)^{1/k} \leqslant (\Sigma|a|^k)^{1/k}+(\Sigma|b|^k)^{1/k} \quad (k\geqslant 1).$$

(2) $$\Sigma|a|^\kappa-\Sigma|b|^\kappa \leqslant \Sigma|a+b|^\kappa \leqslant \Sigma|a|^\kappa+\Sigma|b|^\kappa \quad (0\leqslant\kappa\leqslant 1).$$

In each of (1) and (2) the left-hand inequality reduces to the right-hand one if we replace a by $a+b$ and b by $-b$. The right-hand inequality of (1) is (M). To prove that of (2) it is enough to show that $(a+b)^\kappa \leqslant a^\kappa+b^\kappa$ for $a,\ b\geqslant 0$: this reduces to

$$(1+x)^\kappa \leqslant 1+x^\kappa \quad (x\geqslant 0),$$

which is easily verified by differentiation (since $\kappa-1\leqslant 0$).

We can combine (1) and (2) as follows. For $\lambda>0$ let

$$\alpha=\alpha(\lambda)=\begin{cases}1 & (\lambda\leqslant 1)\\ 1/\lambda & (\lambda\geqslant 1)\end{cases}.$$

Then

(3) $$(\Sigma|a|^\lambda)^\alpha-(\Sigma|b|^\lambda)^\alpha \leqslant (\Sigma|a+b|^\lambda)^\alpha \leqslant (\Sigma|a|^\lambda)^\alpha+(\Sigma|b|^\lambda)^\alpha,$$

or, what is the same thing,

(3′) $$\left|(\Sigma|a+b|^\lambda)^\alpha-(\Sigma|a|^\lambda)^\alpha\right| \leqslant (\Sigma|b|^\lambda)^\alpha.$$

The remaining results in this sub-section involve constant factors. Their value lies in application and the precise values of the constants are without importance. We have first a result roughly equivalent in application to (3′).

$$(4)\qquad \left|\Sigma|a+b|^\lambda-\Sigma|a|^\lambda\right| \leqslant \Sigma|b|^\lambda+R_\lambda(a,\,b)\quad (\lambda>0),$$

where

$$R_\lambda(a,\,b)=\begin{cases}0 & (\lambda\leqslant 1)\\ A_\lambda\Big((\Sigma|a|^\lambda)^{1-1/\lambda}(\Sigma|b|^\lambda)^{1/\lambda}+(\Sigma|a|^\lambda)^{1/\lambda}(\Sigma|b|^\lambda)^{1-1/\lambda}\Big) & (\lambda>1).\end{cases}$$

This is proved [in (2)] if $\lambda\leqslant 1$; suppose then $\lambda>1$. The function

$$\{(1+x)^\lambda-1-x^\lambda\}/(x+x^{\lambda-1})$$

is bounded in $x\geqslant 0$ [consider $x\leqslant\frac{1}{2}$, $\frac{1}{2}\leqslant x\leqslant 2$, $x\geqslant 2$ separately], and non-negative, by Lemma a.

Hence, writing B for A_λ, we have for non-negative a, b,

$$0\leqslant(a+b)^\lambda-a^\lambda-b^\lambda\leqslant B(a^{\lambda-1}b+ab^{\lambda-1}),$$

$$0\leqslant\Sigma(a+b)^\lambda-\Sigma a^\lambda\leqslant\Sigma b^\lambda+B(\Sigma a^{\lambda-1}b+\Sigma ab^{\lambda-1}),$$

while finally

$$\Sigma a^{\lambda-1}b\leqslant(\Sigma a^\lambda)^{1-1/\lambda}(\Sigma b^\lambda)^{1/\lambda},\quad \Sigma ab^{\lambda-1}\leqslant(\Sigma a^\lambda)^{1/\lambda}(\Sigma b^\lambda)^{1-1/\lambda}.$$

The case $a\geqslant 0$, $b\geqslant 0$, is therefore disposed of. Consider now the general case: we have

$$(5)\qquad \Delta=\Sigma|a+b|^\lambda-\Sigma|a|^\lambda\leqslant\Sigma|b|^\lambda+R_\lambda(a,\,b)$$

a fortiori from the positive case. On the other hand, by the same argument,

$$(6)\qquad -\Delta=\Sigma|a^\lambda|-\Sigma|a+b|^\lambda$$

$$=\Sigma|(a+b)+(-b)|^\lambda-\Sigma|a+b|^\lambda\leqslant\Sigma|-b|^\lambda+R_\lambda(a+b,\,-b).$$

If Δ is positive (5) gives us what we want. If, on the other hand, Δ is negative, then $R_\lambda(a+b,\,-b)\leqslant R_\lambda(a,\,b)$, [since $1-1/\lambda>0$] and (6) gives what we want. This completes the proof.

Next we have two simpler results. For $\lambda>0$

$$(7)\qquad \Sigma|a+b|^\lambda\leqslant A_\lambda(\Sigma|a|^\lambda+\Sigma|b|^\lambda),$$

$$(8)\qquad (\Sigma|a+b|^\lambda)^{1/\lambda}\leqslant A_\lambda\{(\Sigma|a|^\lambda)^{1/\lambda}+(\Sigma|b|^\lambda)^{1/\lambda}\},$$

with extensions to more than two sets.

For $\quad |a+b|^\lambda\leqslant\{2\,\mathrm{Max}(|a|,|b|)\}^\lambda\leqslant 2^\lambda(|a|+|b|^\lambda).$

Thus (7) is true with $A_\lambda=2^\lambda$. Further

$$\Sigma|a+b|^\lambda\leqslant 2^\lambda.\,2\,\mathrm{Max}(\Sigma|a|^\lambda,\ \Sigma|b|^\lambda).$$

and so $$(\Sigma|a+b|^{\lambda})^{1/\lambda} \leqslant 2^{1+1/\lambda} \operatorname{Max}\{(\Sigma|a|^{\lambda})^{1/\lambda}, (\Sigma|b|^{\lambda})^{1/\lambda}\}$$
$$\leqslant 2^{1+1/\lambda}\{(\Sigma|a|^{\lambda})^{1/\lambda}+(\Sigma|b|^{\lambda})^{1/\lambda}\},$$
which proves (8).

We now prove that, for $\lambda > 0$,

(9) *if* $\int_E |f-f_n|^{\lambda}\, d\theta \to 0$ *as* $n\to\infty$, *then* $\int_E |f_n|^{\lambda}\, d\theta \to \int_E |f|^{\lambda}\, d\theta$;

(10) *if* $\int_E |f-f_n|^{\lambda}\, d\theta \to 0$, $\int_E |f^*-f_n|^{\lambda}\, d\theta \to 0$, *then* $f^* \equiv f$ *in* E.

In fact, by (3)′ (for integrals),

$$\left|\left(\int|f_n|^{\lambda}\, d\theta\right)^{\alpha} - \left(\int|f|^{\lambda}\, d\theta\right)^{\alpha}\right| \leqslant \left(\int|f_n-f|^{\lambda}\, d\theta\right)^{\alpha} \to 0$$

provided $\int|f|^{\lambda}\, d\theta$ is finite. If the last integral is not finite we can conclude that for any fixed N

$$\int|f_n|^{\lambda}\, d\theta \geqslant \int|[f_n]_N|^{\lambda}\, d\theta \to \int|[f]_N|^{\lambda}\, d\theta$$

and so $\int|f_n|^{\lambda}\, d\theta \to \infty$, since the last expression tends to ∞ with N. Thus (9) is true whether $\int|f|^{\lambda} d\theta$ is finite or infinite.

It is not difficult to deduce (9) also from (4).

For (10) we have $f^*-f = (f_n-f)+(f^*-f_n)$, and so

$$\int|f^*-f|^{\lambda}\, d\theta \leqslant A_{\lambda}\left\{\int|f_n-f|^{\lambda}\, d\theta + \int|f_n-f^*|^{\lambda}\, d\theta\right\}$$

by (7). Since the right-hand side tends to zero the left side is equal to zero, and the non-negative integrand is equivalent to zero.

2.7. *The means* $M_{\mu}(f)$.

2.71. We define, for any finite $\mu \neq 0$,

$$M_{\mu}(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{\mu}\, d\theta\right)^{1/\mu},$$

$$\Lambda_{\mu}(f) = \log M_{\mu}(f).$$

For $\mu = +\infty$ we define $M_{\infty} = e^{\Lambda_{\infty}}$ as the greatest number M such that for every ϵ $|f| > M-\epsilon$ in a set of positive measure, or as ∞ if no M exists. We call this number also $\operatorname{Max}(|f|)$ or the maximum of $|f|$: equivalent functions have the same maximum, and for any f there exists on equivalent $f^*(=[f]_M)$, of which M is the maximum in the ordinary sense. Similarly we define $\operatorname{Min}|f| = M_{-\infty} = e^{\Lambda_{-\infty}}$ as the least m such that $|f| < m + \epsilon$ in a set of positive measure. For a continuous f M and m are, of course, $\text{Max}\,|f|$ and $\text{Min}\,|f|$ in the usual sense. Finally we define

$$\Lambda_0 = \log M_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f|\,d\theta.$$

This integral has a definite value (possibly $+\infty$ or $-\infty$) unless the integral over the positive values of the integrand and that over the negative values are both infinite. In the latter case we regard Λ_0 as taking *all* values from $-\infty$ to $+\infty$, and interpret statements about it in the obvious way (*e.g.* $\Lambda_\mu \geqslant \Lambda_0$ would mean $\Lambda_\mu = +\infty$). $\mu = 0$ is a genuinely exceptional suffix, but the gloss enables us to remove those points of difference that are merely trivial.

We observe that

$$\Lambda_{-\mu}(f) = -\Lambda_\mu(1/f) \quad (-\infty \leqslant \mu \leqslant +\infty), \tag{1}$$

a result which enables us to infer propositions about negative μ from those for positive μ.

2 . 72. We prove next :

$$\Lambda_\mu \to \Lambda_{+\infty} \textit{ as } \mu \to +\infty, \quad \Lambda_\mu \to \Lambda_{-\infty} \textit{ as } \mu \to -\infty. \tag{2}$$

It is enough, by (1), to prove the first. If $M_\infty < +\infty$ there exists an f^* such that

$$|f| \equiv |f^*| \leqslant M_\infty,$$

and so

$$M_\mu(f) = M_\mu(f^*) \leqslant M_\infty.$$

On the other hand, $|f| > M_\infty - \epsilon$ in a set E of measure $\delta > 0$,

$$M_\mu(f) \geqslant \left\{\frac{\delta}{2\pi}(M_\infty - \epsilon)^\mu\right\}^{1/\mu},$$

$$\varliminf_{\mu\to\infty} M_\mu \geqslant M_\infty - \epsilon, \quad \varliminf M_\mu \geqslant M_\infty.$$

Hence $M_\mu \to M_\infty$. If $M_\infty = \infty$ we have, for an arbitrarily large K, $|f| > K$ in a set of positive measure δ,

$$M_\mu \geqslant \left(\frac{\delta}{2\pi}K^\mu\right)^{1/\mu}, \quad \varliminf M_\mu \geqslant K, \quad \lim M_\mu = \infty.$$

2 . 73. We show next : Λ_μ *is an increasing function of* μ *(in the wide sense).* We have to show that $\Lambda_{\mu_2} \geqslant \Lambda_{\mu_1}$ if $\mu_2 > \mu_1$. If $\mu_1 > 0$ this follows from § 2 . 5 (7) [with $|f|^{\mu_1}$ for f, $k = \mu_2/\mu_1$], and (1) above then shows that it is true also for $\mu_2 < 0$. (Incidentally we see that

$$M_{+0} = \lim_{\mu\to+0} M_\mu$$