

# ELASTICITY IN ENGINEERING MECHANICS

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# ELASTICITY IN ENGINEERING MECHANICS

## PREFACE

The material presented is intended to serve as a basis for a critical study of the fundamentals of elasticity and several branches of solid mechanics, including advanced mechanics of materials, theories of plates and shells, composite materials, plasticity theory, finite element, and other numerical methods. Chapter 1 includes, for ready reference, certain mathematic preliminaries. Depending on the background of the reader, this material may be used either as required reading or as reference material. The main content of the book begins with the theory of deformation in Chapter 2. The theory of stress is presented in Chapter 3. The theories of deformation and stress are treated separately to emphasize their independence of one another and also to emphasize their mathematical similarity. By so doing, one can clearly see that these theories depend only on approximations related to modeling of a continuous medium, and that they are independent of material behavior. The theories of deformation and stress are united in Chapter 4 by the introduction of three-dimensional stress-strain-temperature relations (constitutive relations). The major portion of Chapter 4 is devoted to linearly elastic materials. However, a brief discussion of nonlinear constitutive relations is presented in Appendix 4B. Chapters 5 and 6 treat the plane theory of elasticity, in rectangular and polar coordinates, respectively. Chapter 7 presents the three-dimensional problem of prismatic bars subjected

to end loads. Material on thermal stresses is incorporated in a logical manner in the topics of Chapters 4, 5, and 6.

General solutions of elasticity are presented in Chapter 8. Extensive use is made of appendixes for more advanced topics such as complex variables (Appendix 5B) and stress-couple theory (Appendixes 5A and 6A). In addition, in each chapter, examples and problems are given, along with explanatory notes, references, and a bibliography for further study.

As presented, the book is valuable as a text for students and as a reference for practicing engineers/scientists. The material presented here may be used for several different types of courses. For example, a semester course for senior engineering students may include topics from Chapter 2 (Sections 2-1 through 2-16), Chapter 3 (Sections 3-1 through 3-8), Chapter 4 (Sections 4-1 through 4-7 and Sections 4-9 through 4-12), Chapter 5 (Sections 5-1 through 5-7), as much as possible from Chapter 6 (from Section 6-1 through Section 6-6), and considerable problem solving. A quarter course for seniors could cover similar material from Chapters 2 through 5, with less emphasis on the examples and problem solving. A course for first-year graduate students in civil and mechanical engineering and related engineering fields can include Chapters 1 through 6, with selected materials from the appendixes and/or Chapters 7 and 8. A follow-up graduate course can include most of the Appendix material in chapters 2–6, and the topics in Chapters 7 and 8, with specialized topics of interest for further study by individual students.

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# INTRODUCTORY CONCEPTS AND MATHEMATICS

## PART I INTRODUCTION

### 1-1 Trends and Scopes

Papers on the applications of the theory of elasticity to engineering problems form a significant part of the technical literature in solid mechanics. Many of the solutions presented in current papers employ numerical methods and require the use of high-speed digital computers. This trend is expected to continue into the foreseeable future, particularly with the widespread use of microcomputers and minicomputers as well as the increased availability of supercomputers (Londer, 1985). For example, finite element methods have been applied to a wide range of problems such as plane problems, problems of plates and shells, and general three-dimensional problems, including linear and nonlinear behavior, and isotropic and anisotropic materials. Furthermore, through the use of computers, engineers have been able to consider the optimization of large engineering systems (Gallagher and Zienkiewicz, 1973; Kirsch, 1981), such as the space shuttle. In addition, computers have played a powerful role in the fields of computer-aided design (CAD) and computer-aided manufacturing (CAM) (Ellis and Semenov, 1983).

In addition to finite element methods, older techniques such as finite difference methods have found renewed applications in