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FRACTIONAL DIFFERENTIAL EQUATIONS

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To

Tamara, Rekha, and Margarita

Preface

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables.

The concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by Marquis de L'Hôpital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$ when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$?). In his reply, dated 30 September 1695, Leibniz wrote to L'Hôpital as follows: "... *This is an apparent paradox from which, one day, useful consequences will be drawn.* ..."

Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Grünwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. In fact, in his 700-page textbook, entitled "*Traité du Calcul Différentiel et du Calcul Intégral*" (Second edition; Courcier, Paris, 1819), S. F. Lacroix devoted two pages (pp. 409-410) to fractional calculus, showing *eventually* that

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

In addition, of course, to the theories of differential, integral, and integro-differential equations, and special functions of mathematical physics as well as their extensions and generalizations in one and more variables, some of the areas of present-day applications of fractional calculus include Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on.

The first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [643] published in 1974. One of the most recent works on the subject of fractional calculus is the book of Podlubny [682] published in 1999, which deals principally with fractional differential equations. Some of the latest (but certainly not the last) works especially on fractional models of anomalous kinetics of complex processes are the volumes edited by Carpinteri and Mainardi [132] in 1997 and by Hilfer [340] in 2000, and the book by Zaslavsky [915] published in 2005. Indeed, in the meantime, numerous other works (books, edited volumes, and conference proceedings) have also appeared. These include (for example) the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbas and Marichev [729], which was published in Russian in 1987 and in English in 1993, and the book devoted substantially to fractional differential equations by Miller and Ross [603], which was published in 1993. And today there exist at least two international journals which are devoted almost entirely to the subject of fractional calculus: (i) *Journal of Fractional Calculus* and (ii) *Fractional Calculus and Applied Analysis.*

The main objective of this book is to complement the contents of the other books mentioned above. Many new results, obtained recently in the theory of ordinary and partial differential equations, are not specifically reflected in the book. We aim at presenting, in a systematic manner, results including the existence and uniqueness of solutions for the Cauchy Type and Cauchy problems involving nonlinear ordinary fractional differential equations, explicit solutions of linear differential equations and of the corresponding initial-value problems by their reduction to Volterra integral equations and by using operational and compositional methods, applications of the one- and multi-dimensional Laplace, Mellin, and Fourier integral transforms in deriving closed-form solutions of ordinary and partial differential equations, and a theory of the so-called sequential linear fractional differential equations including a generalization of the classical Frobenius method.

This book consists of a total of eight chapters. Chapter 1 (Preliminaries) provides some basic definitions and properties from such topics of Mathematical Analysis as functional spaces, special functions, integral transforms, generalized functions, and so on. The extensive modern-day usages of such special functions as the classical Mittag-Leffler functions and its various extensions, the Wright (or, more precisely, the Fox-Wright) generalization of the relatively more familiar hypergeometric ${}_pF_q$ function, and the Fox H-function in the solutions of ordinary and partial fractional differential equations have indeed motivated a major part of Chapter 1. Chapter 2 (Fractional Integrals and Fractional Derivatives) contains the definitions and some potentially useful properties of several different families of fractional integrals and fractional derivatives. Chapter 1 and Chapter 2, together, are meant to prepare the reader for the understanding of the various mathematical tools and techniques which are developed in the later chapters of this book.

The fundamental existence and uniqueness theorems for *ordinary* fractional differential equations are presented in Chapter 3 with special reference to the Cauchy Type problems. Here, in Chapter 3, we also consider nonlinear and linear fractional differential equations in one-dimensional and vectorial cases. Chapter

4 is devoted to explicit and numerical solutions of fractional differential equations and boundary-value problems associated with them. Our approaches in this chapter are based mainly upon the reduction to Volterra integral equations, upon compositional relations, and upon operational calculus.

In Chapter 5, we investigate the applications of the Laplace, Mellin, and Fourier integral transforms with a view to constructing explicit solutions of linear differential equations involving the Liouville, Caputo, and Riesz fractional derivatives with constant coefficients. Chapter 6 is devoted to a survey of the developments and results in the fields of *partial* fractional differential equations and to the applications of the Laplace and Fourier integral transforms in order to obtain closed-form solutions of the Cauchy Type and Cauchy problems for the fractional diffusion-wave and evolution equations.

Linear differential equations of *sequential* and *non-sequential* fractional order, as well as systems of linear fractional differential equations associated with the Riemann-Liouville and Caputo derivatives, are investigated in Chapter 7, which incidentally also develops an interesting generalization of the classical Frobenius Method for solving fractional differential equations with *variable* coefficients and a direct way to obtain explicit solutions of such types of differential equations with *constant* coefficients. And, while a survey of a variety of applications of fractional differential equations are treated briefly in many chapters of this book (especially in Chapter 7), a review of some important applications involving fractional models is presented systematically in the last chapter of the book (Chapter 8).

At the end of this book, for the convenience of the readers interested in *further* investigations on these and other closely-related topics, we include a rather large and up-to-date Bibliography. We also include a Subject Index.

Operators of fractional integrals and fractional derivatives, which are based essentially upon the familiar Cauchy-Goursat Integral Formula, were considered by (among others) Sonin in 1869, Letnikov in 1868 onwards, and Laurent in 1884. In recent years, many authors have demonstrated the usefulness of such types of fractional calculus operators in obtaining particular solutions of numerous families of homogeneous (as well as nonhomogeneous) linear ordinary and partial differential equations which are associated, for example, with many of the *celebrated* equations of mathematical physics such as (among others) the *Gauss hypergeometric equation*:

$$z(1-z)\frac{d^2w}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{dw}{dz}-\alpha\beta w=0$$

and the relatively more familiar *Bessel equation*:

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)w=0.$$

In the cases of (ordinary as well as partial) differential equations of *higher* orders, which have stemmed naturally from the Gauss hypergeometric equation, the Bessel equation, and their many relatives and extensions, there have been several seemingly independent attempts to present a remarkably large number of scattered results in a unified manner. For developments dealing extensively with such applications of fractional calculus operators in the solution of ordinary and partial differential equations, the interested reader is referred to the numerous recent works cited in the Bibliography.

This book is written primarily for the graduate students and researchers in many different disciplines in the mathematical, physical, and engineering sciences, who are interested not only in learning about the various mathematical tools and techniques used in the theory and widespread applications of fractional differential equations, but also in *further* investigations which emerge naturally from (or which are motivated substantially by) the physical situations modelled mathematically in the book.

Right from the conceptualization of this book project until the finalization of the typescript in this present form, the authors have gratefully received invaluable suggestions and comments from colleagues at many different academic institutions and research centers around the world. Special mention ought to be made of the help and assistance so generously and meticulously provided by Dr. Margarita Rivero, Dr. Blanca Bonilla, Dr. José R. Franco, Ldo. Luís Rodríguez-Germá, all at the Universidad de La Laguna. Special mention should be made also of Engr. Sergio Ortiz-Villajos Eirín and of Dr. Luís Vázquez at the Universidad Complutense de Madrid.

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Contents

Chapter 1

PRELIMINARIES

This chapter is preliminary in character and contains definitions and properties from such topics of *Analysis* as functional spaces, special functions, and integral transforms.

1.1 Spaces of Integrable, Absolutely Continuous, and Continuous Functions

In this section we present definitions of spaces of p-integrable, absolutely continuous, and continuous functions and their weighted modifications. We also give characterizations of those modified spaces which will be used later.

Let $\Omega = [a, b]$ $(-\infty \leqq a < b \leqq \infty)$ be a finite or infinite interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $L_p(a, b)$ $(1 \leqq p \leqq \infty)$ the set of those Lebesgue complex-valued measurable functions f on Ω for which $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p} \quad (1 \leqq p < \infty) \tag{1.1.1}$$

and

$$\|f\|_\infty = \text{ess sup}_{a \leqq x \leqq b} |f(x)|. \tag{1.1.2}$$

Here ess sup$|f(x)|$ is the essential maximum of the function $|f(x)|$ [see, for example, Nikol'skii [628], pp. 12-13)].

We also need the weighted L^p-space with the power weight. Such a space, which we denote by $X_c^p(a, b)$ $(c \in \mathbb{R};\ 1 \leqq p \leqq \infty)$, consists of those complex-valued Lebesgue measurable functions f on (a, b) for which $\|f\|_{X_c^p} < \infty$, with

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} \quad (1 \leqq p < \infty) \tag{1.1.3}$$

and

$$||f||_{X_c^\infty} = \text{ess sup}_{a \leqq x \leqq b}[x^c|f(x)|]. \tag{1.1.4}$$

In particular, when $c = 1/p$, the space $X_c^p(a,b)$ coincides with the $L_p(a,b)$-space: $X_{1/p}^p(a,b) = L_p(a,b)$.

Let now $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval and let $AC[a,b]$ be the space of functions f which are absolutely continuous on $[a,b]$. It is known [see Kolmogorov and Fomin ([434], p. 338)] that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions:

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \quad (\varphi(t) \in L(a,b)), \tag{1.1.5}$$

and therefore an absolutely continuous function $f(x)$ has a summable derivative $f'(x) = \varphi(x)$ almost everywhere on $[a,b]$. Thus (1.1.5) yields

$$\varphi(t) = f'(t) \text{ and } c = f(a). \tag{1.1.6}$$

For $n \in \mathbb{N} := \{1,2,3,\cdots\}$ we denote by $AC^n[a,b]$ the space of complex-valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(x) \in AC[a,b]$:

$$AC^n[a,b] = \left\{ f : [a,b] \to \mathbb{C} \text{ and } (D^{n-1}f)(x)] \in AC[a,b] \quad (D = \frac{d}{dx}) \right\}, \tag{1.1.7}$$

$\mathbb{C}$ being the set of complex numbers. In particular, $AC^1[a,b] = AC[a,b]$.

This space is characterized by the following assertion [see Samko et al. ([729], Lemma 2.4)].

Lemma 1.1 *The space $AC^n[a,b]$ consists of those and only those functions $f(x)$ which can be represented in the form*

$$f(x) = (I_{a+}^n \varphi)(x) + \sum_{k=0}^{n-1} c_k (x-a)^k, \tag{1.1.8}$$

where $\varphi(t) \in L(a,b)$, c_k $(k = 0,1,\cdots, n-1)$ are arbitrary constants, and

$$(I_{a+}^n \varphi)(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \varphi(t) dt. \tag{1.1.9}$$

It follows from (1.1.8) that

$$\varphi(t) = f^{(n)}(t), \quad c_k = \frac{f^{(k)}(a)}{k!} \quad (k = 0,1,\cdots, n-1). \tag{1.1.10}$$

We also use a weighted modification of the space $AC^n[a,b]$ $(n \in \mathbb{N})$, in which the usual derivative $D = d/dx$ is replaced by the so-called δ-derivative, defined by

$$\delta = xD \quad (D = \frac{d}{dx}). \tag{1.1.11}$$