THE HEAT EQUATION

D. V. WIDDER

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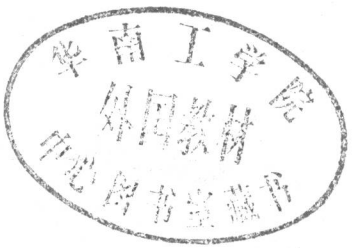

THE HEAT EQUATION

D. V. WIDDER

Department of Mathematics
Harvard University
Cambridge, Massachusetts

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To my son David

PREFACE

This book is designed for students who have had no previous knowledge of the theory of heat conduction nor indeed of the general theory of partial differential equations. On the other hand, a degree of mathematical sophistication is assumed, in that the reader is expected to be familiar with the basic results of the theory of functions of a complex variable, Laplace transform theory, and the standard working tools involving Lebesgue integration. It should be understandable to beginning graduate students or to advanced undergraduates.

The heat equation is derived in Chapters I and XII as a consequence of two basic postulates, easily accepted from physical experience. From this point on, the theorems and results are logical consequences of the heat equation. If the conclusions are at variance with physical facts, and they are slightly so, the fault must be traced to the postulates. For example, the equation forces the conclusion that "action at a distance" is possible. That is, heat introduced at any point of a linear bar raises temperature *instantaneously* at remote portions of the bar. This scandalizes reason and contradicts experiment, so that we must conclude that the postulates are only approximations to the physical situation. But it has also been evident since Fourier's time that they are *good* approximations.

The early chapters develop a theory of the integral transforms that are needed for the integral representations of solutions of the heat equation.

Results that are needed here about the theta-functions of Jacobi are proved in Chapter V. Transforms for which theta-functions are kernels are used for solving boundary-value problems for the finite bar. No previous knowledge of theta-functions is assumed. At only one point is an unproved formula about them employed, and even here, a second approach to the desired result avoids use of that one formula.

Much of the material in Chapters VIII–XIV is based on the author's own research, but it is presented in simplified form. The emphasis is on the expansion of solutions of the heat equation into infinite series. Here the analogies from complex analysis of series developments of analytic functions are very revealing. These are pointed out in detail in Chapter XI. In the final chapter the essential results from four research papers are given simplified proof.

All the material could probably be presented in a half course. More realistically, Chapters V, VI, XII, XIII, and those parts of Chapters VII and VIII dealing with the finite rod, could be omitted. These could be replaced by classic boundary-value problems.

Theorems are generally stated in the same systematic and compact style used by the author in "Advanced Calculus" and in "An Introduction to Transform Theory." The few logical symbols needed to accomplish this are for the most part self-explanatory, but a few are explained parenthetically when introduced.

SYMBOLS AND NOTATION

Page	*Symbol*	*Meaning*
7	$\Rightarrow$	implies
7	$\in$	is a member of
7	$(X \mid P)$	elements X having property P
7	C^n	continuous with derivatives of order $\leqslant n$
8	C	continuous
8	$v_n(x, t)$	heat polynomial
9	[ ]	largest integer $\leqslant$
10	$\rightarrow$	is transformed into
10	$k(x, t)$	source solution
10	$h(x, t)$	derived source solution
10	$l(x, t)$	complementary error function
10	$\theta(x, t)$	theta-function
10	$\varphi(x, t)$	derived theta-function
13, 32	$*$	convolution
14	H	satisfies the heat equation is a temperature function
14	H^*	satisfies the adjoint heat equation
14	$\Leftrightarrow$	implies and is implied by if and only if

16	$'$ (prime)	derivative with respect to *first* variable
19	$\rightarrow$	approaches
25	$O(\ )$	is of the order of
25, 36	$\ll$	is dominated by
26	A	analytic
41	e^{aD}	translation operator
42	e^{aD^2}	Poisson transform operator
43	$\cosh a\sqrt{\mathfrak{D}}$	operator on time variable
43, 250	$e^{a\sqrt{\mathfrak{D}}}$	operator on time variable
44	I	class of functions (real variable)
44	II	class of functions (real variable)
45	III	class of functions (real variable)
60	L	Lebesgue integrable
60	V	bounded variation
68	$o(\ )$	is less than the order of
71	$\uparrow$	nondecreasing
72	$\downarrow$	nonincreasing
107	R	a rectangle
107	Γ	boundary of R
107	γ	sides and base of R
108	γ^*	sides and top of R
122	B	bounded
156	H^{Δ}	difference of positive functions of H
156	H^{o}	Huygens property
170	H_n	Hermite polynomial
170	$\delta_{m,n}$	Kronecker delta
185	$\{m, n\}$	growth of an entire function m, the maximum order; n, the maximum type
198	A^{o}	analytic (restricted)
203	Ap[ ]	Appell transformation
216	$V_n(x, t)$	homogeneous of first kind
217	$h_n(x, t)$	homogeneous of second kind
217	$H_n(x, t)$	homogeneous of second kind
233	$\binom{m}{n}$	binomial coefficient
238	PD	positive definite
247	$\hat{f}$	Fourier transform of f
247	P_a	subclass of frequency functions
250	$f^{(-1/2)}$	Riemann–Liouville fractional integral
251	$\sqrt{\mathfrak{D}} = \mathfrak{D}^{-1/2}$	Riemann–Liouville fractional integral

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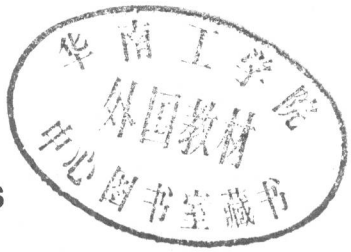

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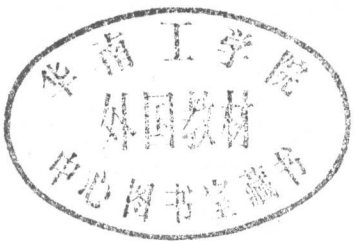
Chapter I

INTRODUCTION



1 INTRODUCTION

In this chapter we shall set down our basic assumptions behind the theory of heat conduction and then derive the heat equation in two dimensions therefrom. We point out its relation to the general theory of partial differential equations and establish a basic Green's formula. We give many examples of solutions of the equation and discuss methods of obtaining new ones. Finally, we set the stage for deeper study by introducing formal definitions and notations.

2 THE PHYSICAL MODEL

In the study of heat, a physical model may be imagined in which heat is considered to be a fluid inside matter, free to flow from one position to another. The amount of fluid present is measured in some unit such as the calorie (cal) or BTU (British Thermal Unit). Evidence of its presence in matter is the temperature thereof, it being assumed that the more heat present the higher the temperature, and that it flows from places of high temperature to places of low temperature. Temperature can be measured

directly by a thermometer; the quantity of heat present is inferred indirectly, as we see by the following definition.

Definition 2 A unit of heat is the amount needed to raise the temperature of one unit of water by one unit of temperature.

For example, in cgs units the unit is called the *calorie* and is the amount of heat necessary to raise one gram of water one degree centigrade. A BTU is the amount to raise one pound one degree Fahrenheit. This is a much larger unit since 1 BTU = 252 cal, approximately.

This definition already makes an assumption about the behavior of heat, for it implies that temperature rise caused by a given quantity of heat is independent of the starting temperature. By experiment this is found to be only approximately true. But in setting up our model we take cognizance of the fact that we are making *approximations* to actual physical conditions.

We now make two postulates about the nature of heat, both of which can be roughly verified by experiment.

Postulate A (Absorption) The amount of increase in the quantity of heat ΔQ in a material is directly proportional to the mass m of the material and to the increase in temperature Δu:

$$\Delta Q = cm\,\Delta u.$$

Here c is the constant of proportionality and is assumed to vary only with the material. It is called the *specific heat* of the material. From Definition 2, $c = 1$ for water. For lead $c = 0.03$ and for silver $c = 0.06$, approximately.

Postulate B (Conduction) Consider a straight bar of homogeneous material, sides insulated, of length Δx and of constant cross section A. If the two ends are held constantly at two different temperatures differing by Δu, the temperature along the bar will vary linearly, and the amount of flow will depend on the abruptness in the change of temperature $\Delta u/\Delta x$. Further, this quantity ΔQ is directly proportional to A, to $\Delta u/\Delta x$, and to Δt, the amount of time elapsed:

$$\Delta Q = -lA\,\frac{\Delta u}{\Delta x}\,\Delta t.$$

Here l is the constant of proportionality, called the *thermal conductivity* of the material, and the flow is in the direction of increasing x when $\Delta u/\Delta x < 0$. To make the postulate graphic, we may think of the bar as extended along the x-axis of an x,u-plane as in Figure 1.

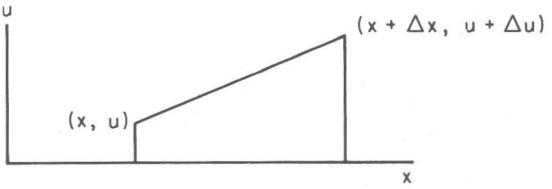


Figure 1

If Δx and Δu are positive, as in the figure, then the flow is opposite to the direction of increasing x and its rate of flow $\Delta Q/\Delta t$ is taken to be negative, $-lA\ \Delta u/\Delta x$. In particular, if the temperature gradient $\Delta u/\Delta x$ is 1° centigrade per centimeter, then l is the number of calories of heat flowing across 1 sq cm of cross section in 1 sec. For water $l = 0.0014$, for lead $l = 0.083$, and for silver $l = 1.0006$, approximately.

If the increase in temperature in a given segment of matter of mass m is a function of time, Postulate A shows that

$$\frac{\partial Q}{\partial t}\bigg|_{t_0} = cm\,\frac{\partial u}{\partial t}\bigg|_{t_0}. \tag{1}$$

That is, the instantaneous rate of increase, at time t_0, of the quantity of heat in the segment is proportional to the instantaneous rate of rise in temperature there.

If the temperature in the above bar is a function of x on the given interval, Postulate B shows that

$$\frac{\partial Q}{\partial t}\bigg|_{x_0} = -lA\,\frac{\partial u}{\partial x}\bigg|_{x_0} \tag{2}$$

at a given instant t_0. That is, the rate of flow across the surface $x = x_0$ is proportional to the temperature gradient there. As pointed out above, the sign of $\partial u/\partial x$ determines the direction of flow (to the left in Figure 1 if $\partial u/\partial x > 0$).

3 THE HEAT EQUATION

The heat equation in two dimensions is the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\,. \tag{1}$$

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