Linear Optimal Control Systems

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To Hélène, Annemarie, and Martin

H. K.

In memory of my parents Yehuda and Tova and to my wife Ilana

R. S.

PREFACE

During the last few years modern linear control theory has advanced rapidly and is now being recognized as a powerful and eminently practical tool for the solution of linear feedback control problems. The main characteristics of modern linear control theory are the state space description of systems, optimization in terms of quadratic performance criteria, and incorporation of Kalman–Bucy optimal state reconstruction theory. The significant advantage of modern linear control theory over the classical theory is its applicability to control problems involving multiinput multioutput systems and time-varying situations; the classical theory is essentially restricted to single-input single-output time-invariant situations.

The use of the term "modern" control theory could suggest a disregard for "classical," or "conventional," control theory, namely, the theory that consists of design methods based upon suitably shaping the transmission and loop gain functions, employing pole-zero techniques. However, we do not share such a disregard; on the contrary, we believe that the classical approach is well-established and proven by practice, and distinguishes itself by a collection of sensible and useful goals and problem formulations.

This book attempts to reconcile modern linear control theory with classical control theory. One of the major concerns of this text is to present design methods, employing modern techniques, for obtaining control systems that stand up to the requirements that have been so well developed in the classical expositions of control theory. Therefore, among other things, an entire chapter is devoted to a description of the analysis of control systems, mostly following the classical lines of thought. In the later chapters of the book, in which modern synthesis methods are developed, the chapter on analysis is recurrently referred to. Furthermore, special attention is paid to subjects that are standard in classical control theory but are frequently overlooked in modern treatments, such as nonzero set point control systems, tracking systems, and control systems that have to cope with constant disturbances. Also, heavy emphasis is placed upon the stochastic nature of control problems because the stochastic aspects are so essential.

We believe that modern and classical control theory can very well be taught simultaneously, since they cover different aspects of the same problems. There is no inherent reason for teaching the classical theory first in undergraduate courses and to defer the modern theory, particularly the stochastic part of it, to graduate courses. In fact, we believe that a modern course should be a blend of classical, modern, and stochastic control theory. This is the approach followed in this book.

The book has been organized as follows. About half of the material, containing most of the analysis and design methods, as well as a large number of examples, is presented in unmarked sections. The finer points, such as conditions for existence, detailed results concerning convergence to steady-state solutions, and asymptotic properties, are dealt with in sections whose titles have been marked with an asterisk. *The unmarked sections have been so written that they form a textbook for a two-semester first course on control theory at the senior or first-year graduate level.* The marked sections consist of supplementary material of a more advanced nature. The control engineer who is interested in applying the material will find most design methods in the unmarked sections but may have to refer to the remaining sections for more detailed information on difficult points.

The following background is assumed. The reader should have had a first course on linear systems or linear circuits and should possess some introductory knowledge of stochastic processes. It is also recommended that the reader have some experience in digital computer programming and that he have access to a computer. We do not believe that it is necessary for the reader to have followed a course on classical control theory before studying the material of this book.

A chapter-by-chapter description of the book follows.

In Chapter 1, "Elements of Linear System Theory," the description of linear systems in terms of their state is the starting point, while transfer matrix and frequency response concepts are derived from the state description. Topics important for the steady-state analysis of linear optimal systems are carefully discussed. They are: controllability, stabilizability, reconstructibility, detectability, and duality. The last two sections of this chapter are devoted to a description of vector stochastic processes, with special emphasis on the representation of stochastic processes as the outputs of linear differential systems driven by white noise. In later chapters this material is extensively employed.

Chapter 2, "Analysis of Control Systems," gives a general description of control problems. Furthermore, it includes a step-by-step analysis of the various aspects of control system performance. Single-input single-output and multivariable control systems are discussed in a unified framework by the use of the concepts of mean square tracking error and mean square input.

Chapter 3, "Optimal Linear State Feedback Control Systems," not only presents the usual exposition of the linear optimal regulator problem but also gives a rather complete survey of the steady-state properties of the Riccati equation and the optimal regulator. It deals with the numerical solution of Riccati equations and treats stochastic optimal regulators, optimal tracking systems, and regulators with constant disturbances and nonzero set points. As a special feature, the asymptotic properties of steady-state control laws and the maximally achievable accuracy of regulators and tracking systems are discussed.

Chapter 4, "Optimal Linear Reconstruction of the State," derives the Kalman–Bucy filter starting with observer theory. Various special cases, such as singular observer problems and problems with colored observation noise, are also treated. The various steady-state and asymptotic properties of optimal observers are reviewed.

In Chapter 5, "Optimal Linear Output Feedback Control Systems," the state feedback controllers of Chapter 3 are connected to the observers of Chapter 4. A heuristic and relatively simple proof of the separation principle is presented based on the innovations concept, which is discussed in Chapter 4. Guidelines are given for the design of various types of output feedback control systems, and a review of the design of reduced-order controllers is included.

In Chapter 6, "Linear Optimal Control Theory for Discrete-Time Systems," the entire theory of Chapters 1 through 5 is repeated in condensed form for linear discrete-time control systems. Special attention is given to state deadbeat and output deadbeat control systems, and to questions concerning the synchronization of the measurements and the control actuation.

Throughout the book important concepts are introduced in definitions, and the main results summarized in the form of theorems. Almost every section concludes with one or more examples, many of which are numerical. These examples serve to clarify the material of the text and, by their physical significance, to emphasize the practical applicability of the results. Most examples are continuations of earlier examples so that a specific problem is developed over several sections or even chapters. Whenever numerical values are used, care has been taken to designate the proper dimensions of the various quantities. To this end, the SI system of units has been employed, which is now being internationally accepted (see, e.g., Barrow, 1966; IEEE Standards Committee, 1970). A complete review of the SI system can be found in the *Recommendations* of the International Organization for Standardization (various dates).

The book contains about 50 problems. They can be divided into two categories: elementary exercises, directly illustrating the material of the text; and supplementary results, extending the material of the text. A few of the problems require the use of a digital computer. The problems marked with an asterisk are not considered to belong to the *textbook* material. Suitable term projects could consist of writing and testing the computer subroutines listed in Section 5.8.

Many references are quoted throughout the book, but no attempt has been made to reach any degree of completeness or to do justice to history. The fact that a particular publication is mentioned simply means that it has been used by us as source material or that related material can be found in it. The references are indicated by the author's name, the year of publication, and a letter indicating which publication is intended (e.g., Miller, 1971b).

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H. K.
R. S.

NOTATION AND SYMBOLS

Chapters are subdivided into sections, which are numbered 1.1, 1.2, 1.3, and so on. Sections may be divided into subsections, which are numbered 1.1.1, 1.1.2, and so on. Theorems, examples, figures, and similar features are numbered consecutively within each chapter, prefixed by the chapter number. The section number is usually given in parentheses if reference is made to an item in another section.

Vectors are denoted by lowercase letters (such as x and u), matrices by uppercase letters (such as A and B) and scalars by lower case Greek letters (such as α and β). It has not been possible to adhere to these rules completely consistently; notable exceptions are t for time, i and j for integers, and so on. The components of vectors are denoted by lowercase Greek letters which correspond as closely as possible to the Latin letter that denotes the vector; thus the n-dimensional vector x has as components the scalars $\xi_1, \xi_2, \cdots, \xi_n$, the m-dimensional vector y has as components the scalars $\eta_1, \eta_2, \cdots, \eta_m$, and so on. Boldface capitals indicate the Laplace or z-transform of the corresponding lowercase time functions [$\mathbf{X}(s)$ for the Laplace transform of $x(t)$, $\mathbf{Y}(z)$ for the z-transform of $y(i)$, etc.].

Operations

x^T	transpose of the vector x
$\text{col}\,(\xi_1, \xi_2, \cdots, \xi_n)$	column vector with components $\xi_1, \xi_2, \cdots, \xi_n$
$(\eta_1, \eta_2, \cdots, \eta_n)$	row vector with components $\eta_1, \eta_2, \cdots, \eta_n$
$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\text{col}\,(x_1, x_2)$	partitioning of a column vector into subvectors x_1 and x_2
$\|x\|$	norm of a vector x
$\dim\,(x)$	dimension of the vector x
A^T	transpose of the matrix A
A^{-1}	inverse of the square matrix A
$\text{tr}\,(A)$	trace of the square matrix A
$\det\,(A)$	determinant of the square matrix A

$\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$	diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \cdots, \lambda_n$
$(e_1, e_2, \cdots, e_n)$	partitioning of a matrix into its columns $e_1, e_2, \cdots, e_n$
$\begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}$	partitioning of a matrix into its rows $f_1, f_2, \cdots, f_n$
$(T_1, T_2, \cdots, T_m)$	partitioning of a matrix into column blocks $T_1, T_2, \cdots, T_n$
$\begin{pmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_m \end{pmatrix}$	partitioning of a matrix into row blocks $U_1, U_2, \cdots, U_m$
$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$	partitioning of a matrix into blocks A, B, C, and D
$\mathrm{diag}(J_1, J_2, \cdots, J_m)$	block diagonal matrix with diagonal blocks $J_1, J_2, \cdots, J_m$
$M > 0$, $M \geq 0$	the real symmetric or Hermitian matrix M is positive-definite or nonnegative-definite, respectively
$M > N$, $M \geq N$	the real symmetric or Hermitian matrix $M - N$ is positive-definite or nonnegative-definite, respectively
$\dot{x}(t)$ or $\frac{dx(t)}{dt}$	time derivative of the time-varying vectoı $x(t)$
$\mathscr{L}\{x(t)\}$	Laplace transform of $x(t)$
$\mathrm{Re}(\alpha)$	real part of the complex number α
$\mathrm{Im}(\alpha)$	imaginary part of the complex number α
$\min(\alpha, \beta)$	the smallest of the numbers α and β
$\min_{\alpha}$	the minimum with respect to α
$\max_{\alpha}$	the maximum with respect to α

Commonly used symbols

0	zero; zero vector; zero matrix
$A(t)$, $A(i)$, A	plant matrix of a finite-dimensional linear differential system

$B(t)$, $B(i)$, B	input matrix of a finite-dimensional linear differential system (B becomes b in the single-input case)
$C(t)$, $C(i)$, C	output matrix of a finite-dimensional linear differential system; output matrix for the observed variable (C becomes c in the single-output case)
$C_e(t)$, $C_e(i)$, $C_{e\infty}$	mean square tracking or regulating error
$C_u(t)$, $C_u(i)$, $C_{u\infty}$	mean square input
$D(t)$, $D(i)$, D	output matrix for the controlled variable (D becomes d in the single-output case)
e	base of the natural logarithm
$e(t)$ or $e(i)$	tracking or regulating error; reconstruction error
e_i	i-th characteristic vector
E	expectation operator
$E(i)$	gain matrix of the direct link of a plant (Ch. 6 only)
f	frequency
$F(t)$, $F(i)$, F, $\bar{F}$	regulator gain matrix (F becomes f in the single-input case)
$G(s)$, $G(z)$	controller transfer matrix (from y to $-u$)
$H(s)$, $H(z)$	plant transfer matrix (from u to y)
i	integer
I	unit matrix
j	$\sqrt{-1}$; integer
$J(s)$, $J(z)$	return difference matrix or function
$K(t)$, $K(i)$, K, $\bar{K}$	observer gain matrix (K becomes k in the single-output case)
$K(s)$	plant transfer matrix (from u to z)
$H_c(s)$, $H_c(z)$	closed-loop transfer matrix
n	dimension of the state x
$N(s)$, $N(z)$	transfer matrix or function from r to u in a control system
P	controllability matrix
$P(t)$, $P(i)$, $\bar{P}$	solution of the regulator Riccati equation
$P(s)$, $P(z)$	controller transfer matrix (from r to u)
P_1	terminal state weighting matrix
Q	reconstructibility matrix
$Q(t)$, $Q(i)$, $\bar{Q}$	variance matrix; solution of the observer Riccati equation
Q_0	initial variance matrix
$Q'(t)$, $Q'(i)$	second-order moment matrix
$r(t)$, $r(i)$	reference variable
$R_1(t)$, $R_1(i)$, R_1	weighting matrix of the state
$R_2(t)$, $R_2(i)$, R_2	weighting matrix of the input

$R_3(t)$, $R_3(i)$, R_3	weighting matrix of the tracking of regulating error
$R_v(t_1, t_2)$, $R_v(t_1 - t_2)$, $R_v(i, j)$, $R_v(i - j)$	covariance function of the stochastic process v
s	variable of the Laplace transform
$S(s)$, $S(z)$	sensitivity matrix or function
t	time
$T(s)$ or $T(z)$	transmission
$u(t)$, $u(i)$	input variable
$v(t)$, $v(i)$	stochastic process
$v_m(t)$, $v_m(i)$	observation noise, measurement noise
v_0	constant disturbance
$v_0(t)$, $v_0(i)$	equivalent disturbance at the controlled variable
$v_p(t)$, $v_p(i)$	disturbance variable
$V(t)$, $V(i)$	intensity of a white noise process
$w(t)$, $w(i)$	white noise process
$W_e(t)$, $W_e(i)$, W_e	weighting matrix of the tracking or regulating error
$W_u(t)$, $W_u(i)$, W_u	weighting matrix of the input
$x(t)$, $x(i)$	state variable
$\hat{x}(t)$, $\hat{x}(i)$	reconstructed state variable
x_0	initial state
$y(t)$, $y(i)$	output variable; observed variable
z	z-transform variable
$z(t)$, $z(i)$	controlled variable
Z	compound matrix of system and adjoint differential equations
$\delta(t)$	delta function
Δ	sampling interval
$\zeta(t)$, $\zeta(i)$	scalar controlled variable
$\eta(t)$, $\eta(i)$	scalar output variable; scalar observed variable
θ	time difference; time constant; normalized angular frequency
λ_i	i-th characteristic value
$\mu(t)$, $\mu(i)$	scalar input variable
$\nu(t)$, $\nu(i)$	scalar stochastic process
ν_i	i-th zero
$\xi(t)$, $\xi(i)$	scalar state variable
π_i	i-th pole
ρ	weighting coefficient of the integrated or mean square input
$\Sigma_v(\omega)$, $\Sigma_v(\theta)$	spectral density matrix of the stochastic process v
$\phi(s)$, $\phi(z)$	characteristic polynomial
$\phi_c(s)$, $\phi_c(z)$	closed-loop characteristic polynomial

$\Phi(t, t_0)$, $\Phi(i, i_0)$	transition matrix
$\psi(s)$, $\psi(z)$	numerator polynomial
ω	angular frequency

SI units

A	ampere
Hz	hertz
kg	kilogram
kmol	kilomole
m	meter
N	newton
rad	radian
s	second
V	volt
Ω	ohm

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CONTENTS

* See the Preface for the significance of the marked sections.