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Linear Differential Operators with Constant Coefficients

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Preface

This book contains a systematic exposition of the facts relating to partial differential equations with constant coefficients. The study of systems of equations in general form occupies a central place. Together with the classical problems of the existence, the uniqueness, and the regularity of the solutions, we also consider the specific problems that arise in connection with overdetermined and underdetermined systems of equations: the extendability of the solutions into a wider region, the extendability of regularity, M-cohomology and so on. Great attention is paid to the connections and the parallels with the theory of functions of several complex variables.

The choice of material was dictated by a number of considerations. Among all the facts relating to general systems of equations, the book contains none that relate to the behavior of differential operators in spaces of slowly growing functions. Missing also are results relating to a single equation in one unknown function: the correctness of the Cauchy problem, certain theorems on p-convexity, and the theory of boundary values, are all set forth in other monographs (Gel'fand and Šilov [3], Hörmander [10] and Treves [4]).

The book consists of two parts. In the first, we set forth the analytic method which forms the basis for the contents of the second part, which itself is dedicated to differential equations. The first part is preceded by an introduction in which the content and methods of Part I are described. All the notes and bibliographical references are collected together in a special section.

This book was written at the suggestion of G. E. Šilov. I am grateful to him for his unfailing support. I have also benefitted greatly by my constant contacts with V. V. Grušin.

Let me make some remarks on how to use the book. For a first reading of § 1 of Chapter I, it is sufficient to limit oneself to the basic definitions. The rest of the content of this section is a detailed foundation for the argument contained in § 2. The content of § 3 of Chapter I, with the exception of the introductory section and 7°, is used only in the second part of the book, beginning with Chapter VII.

Chapter II—IV form an integrated whole. On a first reading of these, one may omit only § 4 of Chapter IV, the content of which is used, in essence, only in Chapter VII. In § 5 of Chapter IV only subsections 1°—3° need be read for an understanding of the remaining material. The general formulation of the fundamental theorem, contained in 4°—6°, is used only in § 14 of Chapter VIII.

In Chapter V, the content of §§ 1—3 represents, in essence, an exposition of more or less known facts from the theory of linear topological spaces, distributions, and Fourier transforms. In Chapter VI, 4° of § 4, 8° of § 5, and § 6 are not connected with the subsequent material, and they may be omitted on a first reading.

In the seventh chapter, §§ 11 and 12 are independent; § 13 of Chapter VIII is of auxiliary value only.

In the first part of the book, the sections are numbered independently in each chapter. In the second part, the sections are numbered sequentially. Formulae are numbered according to the following rule: A formula with the number (a, b) is found in the section with the number b and has the serial number a. The citation of a formula relating to another chapter will contain within the parenthesis the number of the chapter cited, but the citation of a formula within the same section omits the section number.

The symbol □ denotes the end of a proof.

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Part II. Differential Equations with Constant Coefficients

Introduction

The last decade has seen the completion of the foundations for what today we call the general theory of partial differential operators with constant coefficients. The general theory is distinguished from the classical in that the study of particular properties of specific operators has given way to an investigation of the structural properties of operators of general form. Perhaps the best example is provided by the study of the local properties of the solutions of homogeneous equations. Special results on the regularity of the solutions of Laplace's equation, the heat equation and certain others laid the foundation for the singling out of classes of operators having similar properties: elliptic and parabolic. These classes of operators have a common property: Every solution of the corresponding homogeneous equation is infinitely differentiable. The next step was to pose the problem of writing down all differential operators having the same property. Such operators, which we call hypoelliptic, were completely described within the class of operators with constant coefficients.

It was then observed that the regularity of solutions of hypoelliptic equations with constant coefficients is a simple consequence of a more general property which relates to all operators with constant coefficients. This general property consists in the possibility of an exponential representation and, more precisely, consists in the fact that every solution of the corresponding homogeneous equation can be written in the form of an integral with some measure over the set of exponential polynomials satisfying the same equation.

Another line, which today characterizes the general theory, had as its starting point the classical problem of the construction of fundamental solutions in the large. Within the framework of the classical theory, this problem was solved only for certain special types of operators. Then, thanks to the application of distribution theory, the following general result was obtained: For an arbitrary non-zero operator with constant coefficients, there exists in the class of distributions a fundamental solution. Moreover, it turns out that the nonhomogeneous equation corresponding to such an operator is soluble for an arbitrary right side.

The subsequent stage of development along this line was connected with the consideration of a system of equations (with constant coefficients) of general form. In this stage, the raw material for development of the general theory was provided by the corresponding portions of the theory of differential forms and functions of several complex variables. The result of their interaction was the general theorem on the solubility of an arbitrary system of equations with constant coefficients when the right-hand side satisfies the "formal condition of compatibility." This theorem on the solubility of nonhomogeneous systems, together with the already-mentioned theorem on exponential representations, provided, in its turn, a special case of the general theorem on exponential representations, which occupies an essential position in this monograph. A number of other problems in the general theory also lead to this general theorem. The second part of the monograph contains an exposition of the basic branches of the general theory of differential operators with constant coefficients, in the form of a series of consequences of the fundamental theorem on exponential representation.

The proof of the theorem on exponential representation is the main content of the first part of the monograph and also of §4 of Chapter V. In order to help the reader, we now set forth in very brief form some of the simplest cases and some of the ideas of the proof. We presuppose that the reader is familiar with the fundamentals of the theory of distributions.

§ 1. Exponential representation for an ordinary equation with one unknown function

1°. Formulation of the problem. An arbitrary linear differential operator with constant coefficients in R^n will be written in the form

$$p(D)=\sum_{|j|\leqq m} p_j D^j, \qquad p_j\in C,$$

where

$$D^j=i^{|j|}\frac{\partial^{|j|}}{\partial \xi_1^{j_1}\dots\partial \xi_n^{j_n}}, \qquad j=(j_1,\dots,j_n), \quad |j|=j_1+\dots+j_n, \quad i=\sqrt{-1},$$

and $\xi=(\xi_1,\dots,\xi_n)$ is some fixed system of coordinates in R^n. Let $z=(z_1,\dots,z_n)$ be a point of the n-dimensional complex space C^n. The polynomial

$$p(z)=\sum p_j z^j, \qquad z^j=z_1^{j_1}\dots z_n^{j_n}$$

is called the characteristic polynomial of the operator $p(D)$. The algebraic variety $N\subset C^n$, formed by the roots of the polynomial $p(z)$ is also called the characteristic variety.

We choose some domain $\Omega\subset R^n$ and consider the corresponding homogeneous equation

$$p(D)\,u=0, \tag{1.1}$$

in which u is assumed to be a generalized function in Ω. We note that for an arbitrary point z, belonging to the characteristic variety N, the function $\exp(z, -i\xi)$ satisfies Eq. (1). In fact,

$$p(D)\exp(z, -i\xi)=p(z)\exp(z, -i\xi)=0.$$

On the other hand, if the function $\exp(z, -i\xi)$ satisfies (1), then the point z belongs to N. We can also find other exponential polynomials satisfying (1). First of all, we take note of Leibnitz's formula

$$p(D)fg=\sum\frac{1}{j!}D^j f p^{(j)}(D)g, \qquad j!=j_1!\dots j_n!,$$

where $p^{(j)}(z)=D_z^j p(z)$. We apply it to the functions $f=f(\xi)$ and $g=\exp(z, -i\xi)$, where $f(\xi)$ is a polynomial of order α:

$$p(D)f(\xi)\exp(z, -i\xi)=\sum_{|j|\leqq\alpha} D^j f(\xi)\,p^{(j)}(z)\exp(z, -i\xi). \qquad (2.1)$$

It follows from this formula that if at the point $z\in N$ all the derivatives of the polynomial p up to order α vanish, then an arbitrary exponential polynomial of the form $f(\xi)\exp(z, -i\xi)$ satisfies (1). We note that formula (2) implies also that for the exponential polynomial $f(\xi)\exp(z, -i\xi)$ to satisfy (1), it is necessary that $z\in N$[1].

We can now formulate in approximate terms the problem of the exponential representation of the solutions of Eq. (1): To write an arbitrary solution of this equation in the form of an integral with some measure over the set of all exponential polynomials satisfying the same equation. We now solve this problem in the most elementary case.

2°. The case of one independent variable. In this case Eq. (1) is an ordinary equation with constant coefficients, and the characteristic polynomial $p(z)$ has only a finite number of roots $\zeta_1, \dots, \zeta_l$. Let $\alpha_1, \dots, \alpha_l$ be the multiplicity of these roots; as is known, $\sum\alpha_\lambda=m$, where m is the order of the polynomial p. From what we have said earlier, it follows that the exponential polynomial

$$\xi^j\exp(-i\zeta_\lambda\xi), \qquad j=0,\dots,\alpha_\lambda-1, \quad \lambda=1,\dots,l, \qquad (3.1)$$

satisfies (1) on the whole line. We note that an arbitrary exponential polynomial satisfying (1) is a linear combination of the functions (3). Thus, the problem of the exponential representation can be formulated now in these terms: To express an arbitrary solution of (1), defined in the open set Ω, in the form of a linear combination of exponential

1 In Chapter V § 7, we obtain a complete expression for the exponential polynomials satisfying (1).

polynomials of the type (3) [2]. Clearly, if this problem is to be soluble, the domain must be connected. Otherwise, we can construct a solution which is equal to distinct linear combinations of the functions (3) in the different connected components. In view of the fact that the functions (3) are linearly independent in an arbitrary open set, such a solution cannot be represented in the form of a linear combination of functions of the type (3) in the whole set Ω. Therefore, we shall suppose that the region Ω is connected, that is, it is represented as an interval (finite or infinite).

3°. Reduction of the problem. With no essential loss of generality, we shall suppose that the interval Ω has the form $(-a, a)$ for some $a>0$. The symbol $\mathscr{D}(\Omega)$ will denote the space of all complex-valued infinitely differentiable functions on the line, of which the carriers belong to Ω. By $\mathscr{D}_b$ we shall denote for every b, $0<b<a$, the subspace of $\mathscr{D}$ formed by all those functions with carriers belonging to the segment $[-b, b]$. The topology in the space $\mathscr{D}_b$ will be defined by the countable collection of norms

$$\|\phi\|^q = \sum_{|j|\leq q} \sup_{\xi} |D^j \phi(\xi)|.$$

It is clear that $\mathscr{D}(\Omega) = \bigcup_{b<a} \mathscr{D}_b$. The conjugate space $\mathscr{D}'(\Omega)$, that is, the space of distributions over Ω, is, by definition, the collection of all linear functionals on $\mathscr{D}(\Omega)$, having in every subspace $\mathscr{D}_b$ a restriction continuous with respect to the topology of that subspace. A functional $u \in \mathscr{D}'(\Omega)$ satisfies Eq. (1), if

$$\big(u, p^*(D)\,\phi\big) = 0 \tag{4.1}$$

for an arbitrary function $\phi \in \mathscr{D}(\Omega)$. Here $p^*(D)$ is the operator conjugate to $p(D)$; it is equal to the operator $\bar{p}(D)$, where $\bar{p}(z)$ is a polynomial obtained from the polynomial $p(z)$ by replacing each of the coefficients p_j by its complex conjugate.

It is well known that the Fourier transform carries the functions $\phi \in \mathscr{D}_b$ into entire functions, for which all the norms

$$\|\psi\|_q^b = \sup_z (|z|+1)^q \exp(-b\,|\operatorname{Im} z|)\,|\psi(z)|, \qquad q=0, 1, 2, \ldots \tag{5.1}$$

are finite. The space of all entire functions for which all the norms (5) are finite will be denoted by Z^b. We introduce in this space a topology defined by these norms. A well-known theorem states that the Fourier

2 It is well known that the classical Euler theorem on the structure of the solutions of an ordinary equation with constant coefficients provides just such an expression for all classical solutions. On the other hand, every distribution which is a solution of such an equation represents a classical solution and, consequently, the result that we have just reached, is well known. But the purpose of our argument is to provide a very simple example which will illustrate the methods that we shall apply in a much more general situation.

transform sets up a topological isomorphism of the spaces $\mathscr{D}_b$ and Z^b; in other words, the inverse of the Fourier transform is continuous from Z^b to $\mathscr{D}_b$.

The Fourier transform of the generalized function $u \in \mathscr{D}'(\Omega)$ is a linear functional on the space $Z(\Omega) = \bigcup_{b<a} Z^b$, defined by the formula

$$(\tilde{u}, \psi) = (u, \tilde{\psi}), \qquad \psi \in Z(\Omega),$$

where $\tilde{\psi}$ is the inverse Fourier transform of the function ψ. From what we have said above, it follows that the functional $\tilde{u}$ is continuous on every subspace Z^b. From the relation $\widetilde{\bar{p}(z)\,\psi(z)} = \bar{p}(D)\,\tilde{\psi}$ it follows that Eq. (4) can be rewritten as:

$$(\tilde{u}, \bar{p}(z)\,\psi) = 0, \qquad \forall \psi \in Z(\Omega).$$

Thus, in order that the distribution u in Ω should satisfy Eq. (1), it is necessary and sufficient that its Fourier transform should vanish on functions of the form $\bar{p}(z)\,\psi$, where $\psi \in (\Omega)$. The subspace consisting of such functions will be denoted by $\bar{p}\,Z(\Omega)$. We note that the polynomial $\bar{p}$ is connected with p by the relationship $\bar{p}(\bar{z}) = \overline{p(z)}$, from which it follows that for every λ the point $\bar{\zeta}_\lambda$ is a root of the polynomial $\bar{p}$ of multiplicity α_λ.

We now find the Fourier transform of the exponential polynomial (3). We have

$$\begin{aligned}&(\widetilde{\xi^j \exp(-i\,\zeta_\lambda\,\xi)}, \psi)\\ &\quad = (\xi^j \exp(-i\,\zeta_\lambda\,\xi), \tilde{\psi}) = \int \xi^j \exp(-i\,\zeta_\lambda\,\xi)\,\tilde{\psi}(\xi)\,d\xi = c\,D_z^j\,\psi|_{\bar{\zeta}_\lambda},\end{aligned}$$

where c is a constant different from zero. Thus, the Fourier transform of the exponential polynomial $\xi^j \exp(-i\,\zeta_\lambda\,\xi)$ is (up to a constant) the j-th derivative of the delta-function at the point $\bar{\zeta}_\lambda$. So, our task is reduced to the following: To represent every functional on $Z(\Omega)$ that vanishes on the subspace $\bar{p}\,Z(\Omega)$ in the form of a linear combination of delta-functions at the points $\bar{\zeta}_\lambda$ and their derivatives up to certain orders.

4°. The construction of the isomorphism. The problem we have just formulated is essentially like a problem well known in the theory of distributions, namely, to write down the structures of distributions that are concentrated at a point. This analogy suggests the road to the solution. The fundamental step consists in proving that for an arbitrary collection of the values of the derivatives

$$D^j\,\psi(\bar{\zeta}_\lambda), \qquad j = 0, \ldots, \alpha_\lambda - 1, \quad \lambda = 1, \ldots, l,$$

we can always restore the function $\psi \in Z(\Omega)$ and this restoration is unique, up to a function belonging to the subspace $\bar{p}\,Z(\Omega)$.