FLUID DYNAMICS

Dr. J. K. GOYAL ● K. P. GUPTA

PRAGATI

PRAGATI PRAKASHAN

Post Box No. 62, MEERUT—250001

FLUID DYNAMICS

(ADVANCED HYDRO-DYNAMICS)

(FOR HONOURS, POST-GRADUATE AND M. Phil. STUDENTS)

by

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FLUID DYNAMICS
ADVANCED HYDRO-DYNAMICS

नमामीशमीशान निर्वाण रूपं । विभुं व्यापकं ब्रह्म वेदस्वरूपं ।।
निजं निर्गुणं निर्विकल्पं निरीहं । चिदाकाशमाकाशवासं भजेऽहं ।।

We adore you, the guardian of the south-east quarter and Ruler of the whole universe, eternal bliss personified, the omnipresent and all pervading Brahma, manifest in the form of Vedas. We worship **Lord Shiva** *shining in his own glory, devoid of material attributes, undifferentiated, desireless, all pervading consciousness, having nothing to wrap about Himself except ether.*

—**Authors and Publishers**

PREFACE

This book on Fluid Dynamics has been written to cater to the requirements of students studying Fluid Dynamics, Hydro Dynamics or Hydro Mechanics at Post-graduate and Honours standard of all Indian Universities. A large number of articles and problems have been done with the help of Vector Algebra and Vector Calculus. Problems have been selected from standard text books and examination papers of various universities. Students and teachers will find that the subject has been dealt in a lucid and clear style. Enough care has been taken to prevent mistakes and printing errors creeping in, but it is not possible to claim complete immunization. It shall be our pleasure to gratefully acknowledge any suggestions or pointing of errors from teachers and students of the subject.

We are grateful to Shri K. K. Mittal Proprietor M/s Pragati Prakashan and Shri J. P. Rastogi Proprietor Naveen Press, Meerut for the pains they have taken in bringing out the book.

Kanpur
April 1980

J. K. Goyal
and K. P. Gupta

CONTENTS

Chapter **0**

SOME PRELIMINARIES, IMPORTANT FORMULAE

Group A—Results of Vector Analysis

Vectors. Bold face type is used to denote vector and italic letter to denote the magnitude of the vector. For example a is the magnitude of the vector **a**.

(1) If **r** be the position vector of a current point $P(x, y, z)$, then $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, where **i, j, k** are unit vectors along x, y, z axes respectively.

(2) A vector of magnitude one is called **unit vector.**

(3) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$. Then

$$a^2=a_1^2+a_2^2+a_3^2,\quad b^2=b_1^2+b_2^2+b_3^2,$$

$$\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3,$$

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$ Also $\mathbf{a}\times\mathbf{b}=\hat{\mathbf{n}}\,ab\sin\theta$, $\mathbf{a}.\mathbf{b}=ab\cos\theta$

where θ is the angle between **a** and **b** ; and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane containing **a** and **b**.

(4) $\nabla=\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}$, grad $\phi=\nabla\phi$

$$\text{div }\mathbf{a}=\nabla.\mathbf{a}=\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}$$

$$\text{curl }\mathbf{a}=\nabla\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ a_1 & a_2 & a_3\end{vmatrix}$$

$$\text{curl }(\phi\mathbf{a})=\phi\text{ curl }\mathbf{a}+(\text{grad }\phi)\times\mathbf{a}$$

$$\text{div }(\phi\mathbf{a})=\phi\text{ div }\mathbf{a}+(\text{grad }\phi).\mathbf{a}$$

$$\text{div }(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot\text{curl }\mathbf{a}-\mathbf{a}.\text{curl }\mathbf{b}$$

$$\text{curl }(\mathbf{a}\times\mathbf{b})=\mathbf{a}\text{ div }\mathbf{b}-\mathbf{b}\text{ div }\mathbf{a}+(\mathbf{b}.\nabla)\,\mathbf{a}-(\mathbf{a}.\nabla)\,\mathbf{b}$$

grad $(\mathbf{a}.\mathbf{b})=\mathbf{a}\times \text{curl } \mathbf{b}+\mathbf{b}\times \text{curl } \mathbf{a}+(\mathbf{a}.\nabla)\, \mathbf{b}+(\mathbf{b}.\nabla)\, \mathbf{a}$

curl curl $\mathbf{a}=$ grad div $\mathbf{a}-\nabla^2 \mathbf{a}$

$$\mathbf{a}.\nabla=a_1 \frac{\partial}{\partial x}+a_2 \frac{\partial}{\partial y}+a_3 \frac{\partial}{\partial z}$$

div curl $\mathbf{a}=0$, curl grad $\phi=0$ for every $\mathbf{a}$ and ϕ.

(') Let $\mathbf{n}$ be a unit outward normal vector. Then

Gauss Theorem $$\int_S \mathbf{n}.\mathbf{F}\, dS=\int_v \nabla.\mathbf{F}\, dV$$

and $$\int_S \mathbf{n}p\, dS=\int_v \nabla p\, dV.$$

Stoke's Theosem $$\int_c \mathbf{F}.d\mathbf{r}=\int_s \text{curl } \mathbf{F}.\mathbf{n}\, dS.$$

(6) The vector ∇f is normal at every point of the surfacc $f=$const.

Group B—Results of Trigonometry

$$\log(a+ib)=\tfrac{1}{2}\log(a^2+b^2)+i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\log(a-ib)=\tfrac{1}{2}\log(a^2+b^2)-i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\tan^{-1} a-\tan^{-1} b=\tan^{-1}\left(\frac{a-b}{1+ab}\right)$$

$$\tan^{-1} a+\tan^{-1} b=\tan^{-1}\left(\frac{a+b}{1-ab}\right).$$

Group C—Results of Differential Calculus

If ψ is the angle made by a tangent with x-axis, then

$$\tan\psi=\frac{dy}{dx},\ \cos\psi=\frac{dx}{ds},\ \sin\psi=\frac{dy}{ds}.$$

Two lines will be perpendicular if $m_1 m_2=-1$.

$\Gamma(n+1)=n!$ if n is a positive integer.

$\Gamma(1)=1$, $\Gamma(\tfrac{1}{2})=\sqrt{\pi}$, $\Gamma(n)\,\Gamma(1-n)=\pi/\sin n\pi$,

$$\Gamma(n+\tfrac{1}{2})=\frac{\sqrt{\pi}.\Gamma(2n)}{2^{2n-1}}.$$

Group D—Results of Complex Variable

1. The necessary and sufficient condition for a function $f(z)=u+iv$ to be analytic is that (i) u_x, u_y, v_x, v_y all are continuous and (ii) $u_x=v_y$, $u_y=-v_x$ where $u_x=\partial u/\partial x$. (These equations are called Cauchy-Riemann equations.)

2. Cauchy Residue Theorem :—

$$\int_C f(z)\, dz=2\pi i \text{ (Sum of residues within } C).$$

3. Residue at simple pole $z=a$ is the coefficient of $1/(z-a)$ in the expansion of $f(z)$.

Some Important formulae occuring in this book

Chapter 1. Equations of Continuity

1. $\mathbf{q}=-\nabla\phi$ where ϕ is velocity potential.

2. $\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{q}.\nabla=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$

where $\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$.

Flux=Density. Area of the surface. Normal velocity

$\frac{d\rho}{dt}+\rho\nabla.\mathbf{q}=0$: Euler's equation of continuity

$\rho J=\rho_0$: Lagrange's equation of continuity.

$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ is the equation of continuity for liquid.

$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right)F=0$ is the condition for the surface $F(x, y, z, t)=0$ to be a possible form of boundary surface.

$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$: differential equations of stream lines.

$\mathbf{W}=\frac{1}{2}\text{curl } \mathbf{q}=0$ if motion is irrotational.

$ds^2=(dr)^2+(rd\theta)^2+(r\sin\theta\, d\omega)^2$: spherical line element

$ds^2=(dr)^2+(rd\theta)^2+(dz)^2$: cylindrical line element.

Spherical polar co-ordinates : $x=r\sin\theta\cos\omega$, $y=r\sin\theta\sin\omega$, $z=r\cos\theta$.

$$u=-\frac{\partial\phi}{\partial x}=-\frac{\partial\phi}{\partial r},\ v=-\frac{\partial\phi}{\partial y}=-\frac{1}{r}\frac{\partial\phi}{\partial\theta},\ w=-\frac{\partial\phi}{\partial z}=-\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\omega}.$$

Equation of continuity in orthogonal curvilinear co ordinates is $\frac{\partial}{\partial t}(\rho h_1\, da_1.h_2\, da_2.h_3\, da_3)$.

$$=-\Big[h_1\, da_1.\frac{\partial}{h_1\,\partial a_1}(\rho q_1.h_2\, da_2.h_3\, da_3)$$
$$+h_3\, da_2.\frac{\partial}{h_2\,\partial a_2}(\rho q_2.h_1\, da_1.h_3\, da_3)$$
$$+h_3\, da_3.\frac{\partial}{h_3\,\partial a_3}(\rho q_3.h_1\, da_1\, h_2\, da_2)\Big].$$

Chapter 2. Equations of Motion

(1) *Equation of continuity in different forms* :—

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \text{ for liquid}$$

$xv=F(t)$ when motion has spherical symmetry

$xv=F(t)$ when motion has cylindrical symmetry

$\frac{\partial \rho}{\partial t}+\frac{\partial (\rho v)}{\partial x}=0$ when velocity has one component.

(2) $\frac{d\mathbf{q}}{dt}=\mathbf{F}-\frac{1}{\rho}\nabla p$: Euler's equation of motion.

$\mathbf{F}=-\nabla\Omega$. Also $\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=F-\frac{1}{\rho}\frac{\partial p}{\partial x}$: equation of motion when velocity has one component.

Equation of motion for symmetrical motion :

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=F-\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

Bernoulli's pressure equations :

$\int \frac{dp}{\rho}+\frac{1}{2}q^2+\Omega=C$ more steady motion

and $\int \frac{dp}{\rho}-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+\Omega=C$ more for unsteady motion.

Equations for imulsive action : $\mathbf{q}_2-\mathbf{q}_1=\mathbf{I}-\frac{1}{\rho}\nabla\overleftarrow{\omega}.$

Helmholtz vortricity equation : $\frac{d}{dt}\left(\frac{\mathbf{W}}{\rho}\right)=\left(\frac{\mathbf{W}}{\rho}.\nabla\right)\mathbf{q}.$

Chapter 3. Sources, Sinks and Doublets

Lagrange's function ψ :

$$-\frac{\partial \phi}{\partial x}=u=-\frac{\partial \psi}{\partial y}, -\frac{\partial \phi}{\partial y}=v=\frac{\partial \psi}{\partial x}.$$

Complex potential : $W=\phi+i\psi$, $-\frac{dW}{dz}=u-iv$

$$\frac{\partial \phi}{\partial r}=\frac{1}{r}\cdot\frac{\partial \psi}{\partial \theta}, \frac{1}{r}\cdot\frac{\partial \phi}{\partial \theta}=-\frac{\partial \psi}{\partial r}.$$

Complex potential due to source $+m$ at $z=a$ is $W=-m\log(z-a)$, in this case $\phi=-m\log r$, $\psi=-m\theta$.

Complex potential due to doublet μ at $z=a$, angle α is

$$W=\frac{\mu e^{i\alpha}}{z-a}.$$

(i) Image of a source $+m$ w.r.t. a line is a source $+m$ at an equal distance on the other side of the line.

(ii) Image of a doublet μ relative to a line is a doublet μ

at an equal distance on the other side of the line and its axis is antiparallel to the object doublet.

(iii) Image of a source $+m$ w.r.t. circle is a source $+m$ at the inverse point and sink $-m$ at the centre.

(iv) Image of a doublet of strength μ with its axis inclined at an angle α w.r.t. circle is a doublet of strength $\mu' = \mu\, a^2/f^2$ with its axis inclined at an angle $\pi - \alpha$ where a is the radius of circle and f is the distance of doublet from the centre.

Blasius Theorem : $X - iY = \dfrac{i\rho}{2} \int_c \left(\dfrac{dW}{dz}\right)^2 dz.$

Chapter 4. Motion of Cylinders. Part I—Circular cylinders

K. E. of liquid $= -\frac{1}{2}\rho \int \phi \dfrac{\partial \phi}{\partial n}\, ds$ on $r = a$

$$= -\tfrac{1}{2}\,\rho \int_0^{2\pi} \left(\phi\, \frac{\partial \phi}{\partial r}\right)_{r=a} . a\, d\theta.$$

General motion of cylinder

$$\psi = vx - -uy + \frac{\omega}{2}(x^2 + y^2) + c$$

(i) When circular cylinder is in motion with velocity U along x axis, then $W = Ua^2/z$.

(ii) Streaming past a fixed cylinder : $W = Uz + (Ua^2/z)$.

$W = \dfrac{ik}{2\pi} \log z$ due to circulation of strength k only.

Part 2—Elliptic cylinders

Elliptic co-ordinates :

$z = c \cosh \zeta = c \cosh (\xi + i\eta)$, $c^2 = a^2 - b^2$,

$a = c \cosh \alpha$, $b = c \sinh \alpha$, $a + b = c\, e^{\alpha}$.

$T = $ K. E. of liquid $= -\frac{1}{2} \int \left(\phi\, d\psi\right)_{\xi = \alpha}$.

Solution of $\nabla^2 \psi$ is $\psi = e^{-n\xi}(A \cos n\eta + B \sin n\eta)$, $\dfrac{\partial \phi}{\partial \xi} = \dfrac{\partial \psi}{\partial \eta}$.

If an elliptic cylinder is rotating in a liquid, then

$$\phi = \frac{\omega}{4}(a+b)^2 e^{-2\xi} \sin 2\eta, \quad \psi = \frac{\omega}{4}(a+b)^2 e^{-2\xi} \cos 2\eta,$$

$$T = \text{K. E. of liquid} = \frac{\pi}{16}\, \rho\, \omega^2 c^4.$$

K. E. of liquid contained in a rotating elliptic cylinder is