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Lecture Notes in Numerical and Applied Analysis Vol. 6

General Editors: H. Fujita and M. Yamaguti

Recent Topics in
Nonlinear PDE

Edited by

MASAYASU MIMURA (Hiroshima University)
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PREFACE

The meeting on the subject of nonlinear partial differential equations was held at Hiroshima University in February, 1983. Leading and active mathematicians were invited to talk on their current research interests in nonlinear pdes occuring in the areas of fluid dynamics, free boundary problems, population dynamics and mathematical physics. This volume contains the theory of nonlinear pdes and the related topics which have been recently developed in Japan.

Thanks are due to all participants for making the meeting so successful.

Finally, we would like to thank the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan for the financial support.

M. MIMURA
T. NISHIDA

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On the Fluid Dynamical Limit of the Boltzmann Equation

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1. Problem and Results

This paper is a continuation of our paper [16] concerned with the Euler limit of the Boltzmann equation. In [16] we studied the behavior of the density distribution $f^{\varepsilon}(t,x,\xi)$ of rarefied gas particles, when the mean free path $\varepsilon\,(>0)$ tends to zero. More precisely, if the intial density distribution $f_0(x,\xi)$ is sufficiently close to an absolute Maxwellian and satisfies some rather restrictive conditions, then the solution $f^{\varepsilon}(t,x,\xi)$ of the Boltzmann equation with initial data f_0 exists in a time interval $[0,T]$ independent of $\varepsilon \in (0,\infty)$, and f^{ε} converges to a local Maxwellian $f^0(t,x,\xi)$:

$$(1.1)\qquad f^0(t,x,\xi) = \frac{\rho(t,x)}{\{2\pi\theta(t,x)\}^{n/2}}\, e^{-|\xi-v(t,x)|^2/\{2\theta(t,x)\}},$$

when ε tends to zero. Moreover, the fluid dynamic quantities $\{\rho(t,x), v(t,x), \theta(t,x)\}$ (i.e, mass density, flow velocity and temparature) satisfy the compressible Euler equation with initial data specified by $f_0(x,\xi)$. This limiting process is the first approximation to the Hilbert expansion of the solution of the Boltzmann equation.

In this paper we make a more detailed treatment of the Hilbert expansion and establish an asymptotic formula such as

$$f^{\varepsilon}(t,x,\xi) = f^{0}(\varepsilon,t,x,\xi) + \varepsilon f^{1}(\varepsilon,t,x,\xi) + \cdots$$
$$+ \tilde{f}^{0}(\varepsilon,t/\varepsilon,x,\xi) + \varepsilon\tilde{f}^{1}(\varepsilon,t/\varepsilon,x,\xi) + \cdots.$$

In the above formula $f^{j}(\varepsilon,t,x,\xi)$ is sufficiently smooth in $(\varepsilon,t) \in [0,1]\times[0,T]$ $(j=0,1,\cdots)$, and $\tilde{f}^{j}(\varepsilon,\tau,x,\xi)$ is also sufficiently smooth in $(\varepsilon,\tau)\in[0,1]\times[0,T/\varepsilon]$ and behaves like $\exp(-\sigma\tau)$ with $\sigma > 0$ $(j=0,1,\cdots)$. However, the general formula to calculate f^{j} and $\tilde{f}^{j}$ is so complicated that we prove only the special case

$$f^{\varepsilon}(t,x,\xi) = f^{0}(\varepsilon,t,x,\xi) + \tilde{f}^{0}(\varepsilon,t/\varepsilon,x,\xi) + \varepsilon f^{1,*}(\varepsilon,t,x,\xi), \tag{1.2}$$

and suggest the method to prove the next step of the expansion.

The limiting process from the Boltzmann equation to the compressible Euler equation was described in detail in [10] and [16], and we state only the conclusion.

The Cauchy problem of the Boltzmann equation is described as

$$\begin{aligned} &\frac{\partial f}{\partial t} + \xi\cdot\nabla_x f = \frac{1}{\varepsilon} Q[f,f], \qquad t>0,\ (x,\xi) \in R^n \times R^n \quad (n\geq 3), \\ &f|_{t=0} = f_0(x,\xi). \end{aligned} \tag{1.3}$$

Here $f = f(\varepsilon,t,x,\xi)$ is the density distribution of gas particles with the position x and the velocity ξ at time t, $\xi\cdot\nabla_x = \xi_1\partial/\partial x_1 + \cdots + \xi_n\partial/\partial x_n$ and $Q[f,h]$ is the symmetrized collision integral which is a quadratic operator acting on the variable ξ. The scattering potential is assumed to be the cut-off hard type of Grad [5]. $\varepsilon>0$ is the mean free path.

Since we consider (1.3) near an absolute Maxwellian , we put

$$\begin{aligned} &g(\xi) = \rho(2\pi\theta)^{-n/2} e^{-|\xi|^2/(2\theta)}, \qquad \rho > 0,\ \theta > 0, \\ &f(\varepsilon,t,x,\xi) = g + g^{1/2}u(\varepsilon,t,x,\xi), \\ &f_0(x,\xi) = g + g^{1/2} u_0(x,\xi). \end{aligned} \tag{1.4}$$

Then we obtain the equation for the unknown u :

$$\frac{\partial u}{\partial t} = -\xi\cdot\nabla_x u + \frac{1}{\varepsilon} Lu + \frac{1}{\varepsilon}\Gamma[u,u],$$

(1.5) $$u|_{t=0} = u_0(x,\xi),$$

where

$$Lu = 2g^{-1/2}Q[g,g^{1/2}u],$$

$$\Gamma[u,v] = g^{-1/2}Q[g^{1/2}u,\ g^{1/2}v] = \Gamma[v,u],$$

Denoting by $\hat{u}(k,\xi) = F_x u(\cdot,\xi)$ the Fourier transform of u,

$$\hat{u}(k,\xi) = (2\pi)^{-n/2}\int e^{-ik\cdot x}\, u(x,\xi)dx,$$

we convert (1.5) to the following

(1.6)
$$\frac{\partial \hat{u}}{\partial t} = -\ i\xi\cdot k\hat{u} + \frac{1}{\varepsilon}\, L\hat{u} + \frac{1}{\varepsilon}\,\hat{\Gamma}[\hat{u},\hat{u}],\quad i = \sqrt{-1}\ ,$$

$$\hat{u}|_{t=0} = \hat{u}_0(k,\xi),$$

where

(1.7) $$\hat{\Gamma}[u,v](k,\xi) = (2\pi)^{-n/2}\int_{R^n}\Gamma[u(k-k',\cdot),v(k',\cdot)](\xi)dk'\ .$$

The equation (1.6) is actually solved in this paper (see also [16]).

According to [3], the collision integral $Q[f,f]$ has (n+2) invariants $\{h_j(\xi)\ ;\ 0\le j\le n+1\} = \{1,\xi_1,\cdots,\xi_n,\ |\xi|^2/2\}$, i.e,

(1.8) $$\int_{R^n} Q[f,f](\xi)h_j(\xi)d\xi = 0\ ,\quad j = 0,1,\cdots,\ n+1,$$

By the following formula we define fluid dynamic quantities associated with the density $f^{\varepsilon}(t,x,\xi)$ of gas particles, i.e, the mass density $\rho(\varepsilon,t,x)$, fluid flow velocity $v(\varepsilon,t,x)$, internal energy $e(\varepsilon,t,x)$, temparature $\theta(\varepsilon,t,x)$, stress tensor $P(\varepsilon,t,x) = (P_{ij}(\varepsilon,t,x))$, heat flow vector $q(\varepsilon,t,x)$ and pressure $p(\varepsilon,t,x)$ (see [10]):

$$\rho(\varepsilon,t,x) = \int_{R^n} f^{\varepsilon}(t,x,\xi)\ h_0(\xi)\ d\xi\ ,$$

$$\rho(\varepsilon,t,x)v_j(\varepsilon,t,x) = \int f^{\varepsilon}(t,x,\xi)h_j(\xi)d\xi \quad (1\le j\le n)\ ,$$

$$\rho(\varepsilon,t,x)\{e(\varepsilon,t,x) + \frac{1}{2}\ |v(\varepsilon,t,x)|^2\} = \int f^{\varepsilon}(t,x,\xi)h_{n+1}(\xi)d\xi,$$

$$(1.9)\qquad P_{ij}(\varepsilon,t,x) = \int f^{\varepsilon}(t,x,\xi)\{\xi_i - v_i(\varepsilon,t,x)\}\{\xi_j - v_j(\varepsilon,t,x)\}d\xi,$$

$$q_j(\varepsilon,t,x) = \frac{1}{2}\int f^{\varepsilon}(t,x,\xi)|\xi - v(\varepsilon,t,x)|^2\{\xi_j - v_j(\varepsilon,t,x)\}d\xi,$$

$$p(\varepsilon,t,x) = \frac{1}{n}\,\mathrm{tr}\,P(\varepsilon,t,x),$$

$$\theta(\varepsilon,t,x) = \frac{2}{n}\,e(\varepsilon,t,x) = \frac{1}{\rho(\varepsilon,t,x)}\,p\,(\varepsilon,t,x)\ .$$

The last is the ideal gas condition.

Combining (1.3) and (1.8), we obtain

$$(1.10)\qquad \begin{aligned} &\frac{\partial}{\partial t}\rho + \nabla_x\cdot(\rho v) = 0, \\ &\frac{\partial}{\partial t}(\rho v) + \nabla_x\cdot(\rho\, {}^t vv) + \nabla_x\cdot p = 0, \qquad {}^t vv = (v_i v_j)\ , \\ &\frac{\partial}{\partial t}\{\rho(e + \tfrac{1}{2}|v|^2)\} + \nabla_x\cdot\{\rho(e + \tfrac{1}{2}|v|^2) + Pv + q\} = 0\ . \end{aligned}$$

Since $P = pI$ (I = the identity matrix) and $q = 0$ for the local Maxwellian f, the equation (1.10) reduces to the compressible Euler equation for $\varepsilon = 0$ (and $t > 0$).

According to Proposition 3.1 of [12], we have

$$P_{ij}(\varepsilon,\cdot) - p(\varepsilon,\cdot)\delta_{ij} = -2\varepsilon\mu(\theta)\{\tfrac{1}{2}(\frac{\partial}{\partial x_i}v_j + \frac{\partial}{\partial x_j}v_i) - \frac{1}{n}\nabla_x\cdot v\} + O(\varepsilon^2),$$

$$q_j(\varepsilon,\cdot) = -\,\varepsilon\kappa(\theta)\,\frac{\partial}{\partial x_j}\theta + O(\varepsilon^2)\ ,$$

Thus the fluid dynamic quatitites $\{\rho, v, \theta\}$ obtained from the density $f(\varepsilon,t,x,\xi)$ given in Theorem 1.2 will satisfy the compressible Navier-Stokes equation with the error of $O(\varepsilon^2)$. A more delicate treatment will be given elsewhere.

To solve the equation (1.6) we use several function spaces and norms (cf. [16]).

We introduce these spaces. All functions are measurable or continuous.

$$(1.11)\qquad X^{\alpha}_{\ell,\beta} \ni u(k,\xi) \iff$$

$$|u|_{\alpha,\ell,\beta} = \sup_{k,\xi\ R^n} e^{\alpha(1+|k|)}(1+|k|)^{\ell}(1+|\xi|)^{\beta}|u(k,\xi)| < \infty\ .$$

(1.12) $\dot{X}^{\alpha}_{\ell,\beta} \ni u(k,\xi) \iff$

$u \in X^{\alpha}_{\ell,\beta}$ and $|\chi(|k|+|\xi|>R)u|_{\alpha,\ell,\beta} \to 0 \ (R \to \infty)$.

Here $\chi(|k|+|\xi|>R)$ is the characteristic function of the set $\{(k,\xi)\in R^n\times R^n; |k|+|\xi|>R\}$. With a Banach space X, $B^0(D;X)$ denotes the space of X-valued, bounded and continuous functions defined on D. We put $R_T = [0,1]\times[0,T]$, $R_T^* = R_T \setminus \{(0,0)\}$ and $\tilde{R}_T = \{(\varepsilon,\tau);\ (\varepsilon,\varepsilon\tau)\in R_T\}$. For $m = (m_1,m_2)$, $m_1 \geq 0$, $m_2 \geq 0$ (resp. $m \geq 0$),

(1.13) $Z^{m,\alpha,\gamma}_{\ell,\beta,T} = B^{m,\gamma}(R_T;\dot{X}^{\alpha}_{\ell,\beta}) \ni u = u(\varepsilon,t) \iff$

$$e^{(\alpha-\gamma t)(1+|k|)}\left(\frac{\partial}{\partial\varepsilon}\right)^i\left(\frac{\partial}{\partial t}\right)^j u(\varepsilon,t) \in B^0(R_T;\dot{X}^0_{\ell-i-j,\beta-i-j})$$

for $0 \leq i \leq m_1$ and $0 \leq j \leq m_2$ (resp. $0 \leq i+j \leq m$).

$$\| u \|_{m,\alpha,\gamma,\ell,\beta,T} = \sum_{i,j} \sup_{0\leq\varepsilon\leq1,0\leq t\leq T} \left|\left(\frac{\partial}{\partial\varepsilon}\right)^i\left(\frac{\partial}{\partial t}\right)^j u(\varepsilon,t)\right|_{\alpha-\gamma t,\ell-i-j,\beta-i-j}.$$

(1.14) $Z^{m,\alpha,\gamma,*}_{\ell,\beta,T} \equiv B^{m,\gamma}(R^*_T;X^{\alpha}_{\ell,\beta})$ is defined similarly.

(1.15) $Z^{m,\alpha,\gamma,\sigma}_{\ell,\beta,T} \equiv B^{m,\gamma,\sigma}(\tilde{R}_T;\dot{X}^{\alpha}_{\ell,\beta}) \ni u(\varepsilon,\tau) \iff$

$$e^{(\alpha-\varepsilon\gamma\tau)(1+|k|)}e^{\sigma\tau}\left(\frac{\partial}{\partial\varepsilon}\right)^i\left(\frac{\partial}{\partial\tau}\right)^j u(\varepsilon,\tau) \in B^0(\tilde{R}_T;\dot{X}^0_{\ell-i-j,\beta-i-j}),$$

for $0 \leq i \leq m_1$ and $0 \leq j \leq m_2$ (resp. $0 \leq i+j \leq m$).

$$\| u \|_{m,\alpha,\gamma,\sigma,\ell,\beta,T} = \sum_{i,j} \sup_{0\leq\varepsilon\leq1,0\leq\varepsilon\tau\leq T} e^{\sigma\tau} \times \left|\left(\frac{\partial}{\partial\varepsilon}\right)^i\left(\frac{\partial}{\partial\tau}\right)^j u(\varepsilon,\tau)\right|_{\alpha-\varepsilon\gamma t,\ell-i-j,\beta-i-j} .$$

Now we can state our results (cf. Theorem 1.1 of [16]).

Theorem 1.1. *Let* g *be an absolute Maxwellian and let* $\alpha > 0$, $\ell > n+1$, $\beta \geq 1$.

Then there exist positive numbers $a_1, b_0, \tilde{b}_0$ *and* b_1^* *such that for each initial data* $f_0 = g + g^{1/2}u_0$ *satisfying*

$$\hat{u}_0 \in \dot{X}^{\alpha}_{\ell,\beta}, \qquad |\hat{u}_0|_{\alpha,\ell,\beta} \le a_1 ,$$

the following statements hold with constants $\gamma > 0$, $T > 0$ $(\alpha - \gamma T \ge 0)$ *and* $\sigma > 0$.

(i) For each $\varepsilon \in (0,1]$, (1.3) *(resp.* (1.6)) *has a unique solution* $f(\varepsilon,t,x,\xi)$ *(resp.* $\hat{u}(\varepsilon,t,k,\xi)$) *on the time interval* $[0,T]$, *and there hold*

$$f = g + g^{1/2}u,$$

$$u(\varepsilon,t) = u^0(\varepsilon,t) + \tilde{u}^0(\varepsilon,t/\varepsilon) + \varepsilon u^{1,*}(\varepsilon,t)$$

$$\hat{u}^0(\varepsilon,t) \in Z^{(0,1),\alpha,\gamma,}_{\ell,\beta,T}, \quad \| \hat{u}^0 \|_{(0,1),\alpha,\gamma,\ell,\beta,T} \le b_0 \, |\hat{u}_0|_{\alpha,\ell,\beta},$$

$$\hat{\tilde{u}}^0(\varepsilon,\tau) \in Z^{(0,1),\alpha,\gamma,\sigma}_{\ell,\beta,T}, \quad \| \hat{\tilde{u}}^0 \|_{(0,1),\alpha,\gamma,\sigma,\ell,\beta,T} \le \tilde{b}_0 |\hat{u}_0|_{\alpha,\ell,\beta},$$

$$\varepsilon\hat{u}^{1,*}(\varepsilon,t) \in Z^{(0,1),\alpha,\gamma,*}_{\ell,\beta,T}, \quad \| \hat{u}^{1,*} \|_{(0,0),\alpha,\gamma,\ell,\beta,T} \le b_1^* \, |\hat{u}_0|_{\alpha,\ell,\beta} .$$

(ii) For $t \in (0,T]$, $f(0,t,x,\xi) = g(\xi) + g(\xi)^{1/2}u^0(0,t,x,\xi)$ *is a local Maxwellian whose fluid dynamical quantities* $\{\rho, v, \theta\}$ *are the solution of the compressible Euler equation* (1.10) *with* $P = pI$ *and* $q = 0$.

(iii) Moreover, there hold

$$\hat{u}^0 \in Z^{1,\alpha,\gamma}_{\ell,\beta,T}, \quad \| \hat{u}^0 \|_{1,\alpha,\gamma,\ell,\beta,T} \le b_0^1 \, |\hat{u}_0|_{\alpha,\ell,\beta},$$

$$\hat{\tilde{u}}_0 \in Z^{1,\alpha,\gamma,\sigma}_{\ell,\beta,T}, \quad \| \hat{\tilde{u}}^0 \|_{1,\alpha,\gamma,\sigma,\ell,\beta,T} \le \tilde{b}_0^1 \, |\hat{u}_0|_{\alpha,\ell,\beta},$$

$$\varepsilon\hat{u}^{1,*} \in Z^{1,\alpha,\gamma,*}_{\ell,\beta,T}, \quad \text{and} \quad \varepsilon\frac{\partial}{\partial\varepsilon}\hat{u}^{1,*}\Big|_{\varepsilon=0} = 0 \ (t>0),$$

Theorem 1.2. *Let* g *be an absolite Maxwellian and let*

$$\alpha > 0, \ \ell > n+3, \ \beta \ge 2.$$

Then there exist positive numbers $a_2, b_j, \tilde{b}_j$ $(j=0,1)$, b_2^*, γ_j, α_j $(j=0,1,\ \alpha_0 = \alpha,\ \alpha - \gamma_0 T \ge \alpha_1,\ \alpha_1 - \gamma_1 T \ge 0)$ *and* σ *such that for each initial data* $f_0 = g + g^{1/2}u_0$ *satisfying*

$$\hat{u}_0 \in \dot{X}^{\alpha}_{\ell,\beta}, \quad |\hat{u}_0|_{\alpha,\ell,\beta} \le a_2 ,$$

the solution $f(\varepsilon,t,x,\xi)$ *of* (1.3) (*resp.* $\hat{u}(\varepsilon,t,k,\xi)$ *of* (1.6)) *is described in the following formula*

$$u(\varepsilon,t) = u^0(\varepsilon,t) + \tilde{u}^0(\varepsilon,t/\varepsilon) + \varepsilon u^1(\varepsilon,t) + \varepsilon\tilde{u}^1(\varepsilon, t/\varepsilon) + \varepsilon^2 u^{2,*}(\varepsilon,t),$$

$$\hat{u}^j(\varepsilon,t) \in Z^{2,\alpha_j,\gamma_j}_{\ell-j,\beta,T}, \quad \| u^j \|_{2,\alpha_j,\gamma_j,\ell-j,\beta,T} \le b_j |\hat{u}_0|_{\alpha,\ell,\beta},$$

$$\hat{\tilde{u}}^j(\varepsilon,\tau) \in Z^{2,\alpha_j,\gamma_j,\sigma}_{\ell-j,\beta,T}, \quad \| \hat{\tilde{u}}^j \|_{2,\alpha_j,\gamma_j,\sigma,\ell-j,\beta,T} \le b_j |\hat{u}_0|_{\alpha,\ell,\beta},$$

for $j = 0,1$, *and*

$$\varepsilon^2 \hat{u}^{2,*}(\varepsilon,t) \in Z^{2,\alpha_1,\gamma_1,*}_{\ell,\beta,T}, \quad \| \hat{u}^{2,*} \|_{1,\alpha_1,\gamma_1,\ell-1,\beta,T} \le b_2^* |\hat{u}_0|_{\alpha,\ell,\beta},$$

$$\varepsilon\frac{\partial}{\partial\varepsilon} \hat{u}^{2,*}(\varepsilon,t)|_{\varepsilon=0} = 0, \quad \varepsilon^2\frac{\partial^2}{\partial\varepsilon^2} \hat{u}^{2,*}|_{\varepsilon=0} = 0 \quad (t > 0).$$

We note that if $\hat{u} \in X^{\alpha}_{\ell,\beta}$, then $u(x,\xi)$ is analytic in $x \in R^n + iB_\alpha$, $B_\alpha = \{y \in R^n \,;\, |y| < \alpha\}$, and uniformly bounded on $R^n + i\bar{B}_\delta$, $0 < \delta < \alpha$. According to the results of Theorem 1.1, we put

$$
\begin{aligned}
&f^0(\varepsilon,t,x,\xi) = g(\xi) + g(\xi)^{1/2}u^0(\varepsilon,t,x,\xi), \\
(1.16)\qquad &\tilde{f}^0(\varepsilon,t,/\varepsilon,x,\xi) = g(\xi)^{1/2}\,\tilde{u}^0(\varepsilon,t/\varepsilon,x,\xi), \\
&f^{1,*}(\varepsilon,t,x,\xi) = g(\xi)^{1/2}u^{1,*}(\varepsilon,t,x,\xi).
\end{aligned}
$$

Then we have the desired formula (1.2).

Similar expansion formula can be estableshed using the results of Theorem 1.2.

Considering that f^0, $\tilde{f}^0$ and $f^{1,*}$ are analytic in $x \in R^n + iB_{\alpha-\gamma t}$ for $0 < t < T$, our existence theorem is of Cauchy-Kowalewski type ([8],[9]). We hope to find more natural existence theorems.

2. Some estimates

Denoting the unknown by $u(k,\xi)$ instead of $\hat{u}(k,\xi)$, we write (1.6) as

$$
(2.1)\qquad
\begin{aligned}
&\frac{\partial u}{\partial t} = - i\xi\cdot ku + \frac{1}{\varepsilon} Lu + \frac{1}{\varepsilon}\hat{\Gamma}[u,u], \\
&u|_{t=0} = u_0(k,\xi).
\end{aligned}
$$

We define the linearized Boltzmann operator

(2.2) $$B(k) = -i\xi\cdot k + L.$$

Then the equation (2.1) reduces to

(2.3) $$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon} B(\varepsilon k)u + \frac{1}{\varepsilon}\hat{\Gamma}[u,u], \qquad u|_{t=0} = u_0(k,\xi).$$

The operator $B(k)$ acts on the variable ξ with the parameter $k \in R^n$. $B(k)$ generates a strongly continuous semi-group $e^{tB(k)}$ in various function spaces on R^n_ξ, for example in $\dot{L}^\infty_\beta$, where

(2.4) $$L^\infty_\beta = \{f(\xi)\ ;\ (1+|\xi|)^\beta f(\xi) \text{ is measurable and bounded }\},$$
$$\dot{L}^\infty_\beta = \{f \in L^\infty_\beta\ ;\ (1+|\xi|)^\beta |f(\xi)| \to 0 \text{ uniformaly as } |\xi| \to \infty\},$$

with the norm

(2.5) $$|f|_\beta = \sup_\xi (1+|\xi|)^\beta |f(\xi)| .$$

Thus the equation (2.3) can be rewritten as the integral equation

(2.7) $$u(t) = e^{tB(\varepsilon k)/\varepsilon}u_0(k,\cdot) + \int_0^t e^{(t-s)B(\varepsilon k)/\varepsilon}\frac{1}{\varepsilon}\hat{\Gamma}[u(s),u(s)](k,\cdot)ds.$$

Now we quote some fundamental properties of L and Γ ([5],[6]). We denote by $c(\lambda), d(\beta), \cdots$ the constants ≥ 0 depending on the parameters λ, β, $\cdots\cdots$.

Lemma 2.1 *(i) The operator* L *has the decomposition*

$$L = -\Lambda + K,$$

Λ *is a multiplication operator,* $\Lambda = \nu(\xi)\times$, *and* K *is an integral operator in* ξ. *Moreover*

(2.8) $\nu(\xi)$ *is contiouous and* $\nu_0 \leq \nu(\xi) \leq \nu_1(1+|\xi|)$

with positive constants ν_0 *and* ν_1 , *and with a constant* $c(\beta) \geq 0$

(2.9) $$|Ku|_\beta \leq c(\beta)|u|_{\beta-1}, \quad \beta \in R.$$

(ii) The speatrum $\sigma(L)$ *of* L *is invariant in* L^∞_β *and* $\dot{L}^\infty_\beta$, $\beta \in R$, *and contained in* $(-\infty,0]$. L *has* 0 *as an isolatied eigenvalue of multiplicity* $n+2$. *Denoting the corresponding eigenprojection by* $P(0)$ ($= \Sigma P_j(0)$, *see Lemma 2.2. (i)(c)), we have*

(2.10) $$P(0)\Gamma[u,v] = 0 , \quad u,v \in L^\infty_\beta\ (\beta \geq 0),$$

(2.11) $|P(0)u|_\beta \le c(\beta,\beta')|u|_{\beta'}$ for any $\beta,\beta' \in R$.

(iii) The operator $\Lambda^{-1}\Gamma[\ ,\ ]$ *is a continuous mapping from* $L^\infty_\beta \times L^\infty_\beta$ *(resp.* $\dot{L}^\infty_\beta \times \dot{L}^\infty_\beta$ *) to* L^∞_β *(resp.* $\dot{L}^\infty_\beta$ *) for* $\beta > 0$, i.e,

(2.12) $|\Lambda^{-1}\Gamma[u,v]|_\beta \le d(\beta)|u|_\beta|v|_\beta$. $\beta \ge 0$.

The following Lemma is concerned with the spectral properties of $B(k)$, essentially due to Ellis-Pinsky [4], and crucial in the study of the Boltzmann equation (e.g, [11], [14], [15] and [16]).

Lemma 2.2. *(i) There is a positive number* κ_0 *such that for* $|k| \le \kappa_0$ $B(k)$ *has* $(n+2)$ *eigenvalues* $\lambda_j(k)$ $(j=0,\cdots, n+1)$ *and coresponding eigen-projections* $P_j(k)$ *of rank* 1 *satisfying the following (a),(b) and (c).*

(a) $B(k)P_j(k) = \lambda_j(k)P_j(k)$, $j=0,1,\cdots, n+1$, $|k| \le \kappa_0$.

$\lambda_j \in C^\infty(\overline{B}_{\kappa_0})$, $\mathrm{Re}\,\lambda_j(k) \le 0$ *and*

$\lambda_j(k) = \pm i\lambda^{(1)}_j|k| - \lambda^{(2)}_j|k|^2 + O(|k|^3)$ $(|k| \to 0)$

with the coefficients $\lambda^{(1)}_j \in R$ *and* $\lambda^{(2)}_j > 0$.

(b) $P_j(k) \in C^\infty(\overline{B}_{\kappa_0})$, *and there exists a constant* $c_j(\beta,\beta')$ *such that*

$$|P_j(k)u|_\beta \le c_j(\beta,\beta')|u|_{\beta'}, \qquad (\beta,\beta' \in R).$$

(c) Put $P(k) = \Sigma P_j(k)$. *Then* $\sigma(B(k)(1-P(k))) \subset \{\lambda\ ;\ \mathrm{Re}\,\lambda < -\sigma_0\}$. *with some* $\sigma_0 > 0$.

$P(0) = \Sigma P_j(0)$ *is the eigenprojection in* (2.10).

(c) If $|k| \ge \kappa_0$, $\sigma(B(k)) \subset \{\lambda\ ;\ \mathrm{Re}\ \lambda < -\sigma_0\}$.

(ii) Let $u = u(\xi) \in \dot{L}^\infty_\beta$ *(resp.* $u = u(k,\xi) \in \dot{X}^\alpha_{\ell,\beta}$ *). Then*

$$e^{tB(k)}u \in B^0([0,\infty)\times R^n_k\ ;\ \dot{L}^\infty_\beta)$$

$$(\textit{resp. } e^{tB(k)}u \in B^0([0,\infty)\ ;\ \dot{X}^\alpha_{\ell,\beta})).$$

Let $\chi(k) \in C^\infty_0(R^n_k)$, $0 \le \chi(k) \le 1$, $\chi(k) = 0$ *for* $|k| \ge \kappa_0$, $= 1$ *for* $|k| \le \kappa_0/2$, *and* $Q(k) = \{1-P(k)\}\chi(k) + \{1 - \chi(k)\}$. *Then there hold*

$$|e^{tB(k)}P(k)u|_\beta \le e(\beta,\beta')|u|_{\beta'} \qquad (\beta,\beta' \in R),$$

$$|e^{tB(k)}Q(k)u|_\beta < g(\beta)e^{-\sigma_0 t}|u|_\beta \quad (\beta \in R),$$

with constants $e(\beta,\beta')$ *and* $g(\beta) \ge 0$.

The following lemmas are simple consequences of the above.

Lemma 2.3. *Let* $\alpha \ge 0$, $\ell > n$ *and* $\beta \ge 0$. *Then* $\Lambda^{-1}\Gamma[\ ,\ ]$ *and* $B(\epsilon k)^{-1}Q(\epsilon k)\hat{\Gamma}[\ ,\ ]$ *are continuous mappings from* $\dot{X}^\alpha_{\ell,\beta} \times \dot{X}^\alpha_{\ell,\beta}$ *to* $\dot{X}^\alpha_{\ell,\beta}$. *Moreover*

$$(2.13) \qquad |\Lambda^{-1}\hat{\Gamma}[u,v],|_{\alpha,\ell,\beta} \le d(\ell,\beta)\,|u|_{\alpha,\ell,\beta}\,|v|_{\alpha,\ell,\beta},$$

$$(2.14) \qquad |B(\epsilon k)^{-1}Q(\epsilon k)\,\hat{\Gamma}[u,v]\,|_{\alpha,\ell,\beta} \le d(\ell,\beta)\,|u|_{\alpha,\ell,\beta}|v|_{\alpha,\ell,\beta},\ \epsilon \ge 0,$$

with a constant $d(\ell,\beta) \ge 0$.

Lemma 2.4. *Define the functions* $\mu_j(k,\epsilon k)$ *and* $P_j^1(k,\epsilon k)$ *by*

$$(2.15) \qquad \mu_j(k,\epsilon k) = \frac{1}{\epsilon}\lambda_j(\epsilon k) = \int_0^1 k\cdot\nabla_k\lambda_j(\theta\epsilon k)d\theta,$$

$$(2.16) \qquad P_j^1(k,\epsilon k) = \frac{1}{\epsilon}\{P_j(\epsilon k) - P_j(0)\} = \int_0^1 k\cdot\nabla_k P_j(\theta\epsilon k)d\theta.$$

Then both of $\mu_j(k,\epsilon k)$ *and* $P_j^1(k,\epsilon k)$ *are in* $B^\infty([0,1] \times \bar{B}_{\kappa_0})$. *Moreover*

$$(2.17) \qquad |(\tfrac{\partial}{\partial\epsilon})^i\mu_j(k,\epsilon k)| \le c_i\,|k|^{i+1},\ i=0,1,\cdots,\ 0 \le j \le n+1,$$

$$(2.18) \qquad |(\tfrac{\partial}{\partial\epsilon})^i P_j^1(k,\epsilon k)u|_\beta \le c_{i,j}(\beta,\beta')\,|k|^{i+1}|u|_{\beta'},\ i=0,1,\cdots,\ 0 \le j \le n+1,$$

for $\beta,\beta' \in R$.

In the proof of (2.14), we note that if we define the multiplication operator $A(k) = -i\xi\cdot k - \nu(\xi)$, then $A(\epsilon k)^{-1}\Lambda$ is a bounded operator in $\dot{L}^\infty_\beta$ and $\dot{X}^\alpha_{\ell,\beta}$ with the bound 1. Thus the equality

$$B(\epsilon k)^{-1}\Lambda = A(\epsilon k)^{-1}\Lambda - B(\epsilon k)^{-1}KA(\epsilon k)^{-1}\Lambda$$

shows (2.14), because $Q(\epsilon k)$ and $Q(\epsilon k)B(\epsilon k)^{-1} = B(\epsilon k)^{-1}Q(\epsilon k)$ are uniformly bounded in $\dot{L}^\infty_\beta$ with respect to $\epsilon \ge 0$ and $k \in R^n$.

Now we treat the terms appearing in (2.7). First, noting the equality $1 = P(\varepsilon k)\chi(\varepsilon k) + Q(\varepsilon k)$, we have

$$(2.19)\qquad e^{tB(\varepsilon k)/\varepsilon} = \sum_{j=0}^{n+1} e^{-t\mu_j(k,\,\varepsilon k)} P_j(\varepsilon k)\chi(\varepsilon k) + e^{tB(\varepsilon k)/\varepsilon}Q(\varepsilon k)$$

$$\equiv \sum_{j=0}^{n+1} F_j(t,k,\varepsilon k) + G(t/\varepsilon,k,\varepsilon k) \equiv F + G\ .$$

Next, noting (2.10) and the corresponding equality $P(0)\hat{\Gamma} = 0$, we have

$$(2.20)\qquad \int_0^t e^{(t-s)B(\varepsilon k)/\varepsilon}\,\frac{1}{\varepsilon}\,\Gamma[u(s),u(s)]ds$$

$$= \int_0^t \sum_{j=0}^{n+1} e^{-(t-s)\mu_j(k,\varepsilon k)} P_j^{\,1}(k,\varepsilon k)\chi(\varepsilon k)\hat{\Gamma}[u(s),u(s)]ds$$

$$+ \frac{1}{\varepsilon}\int_0^t e^{(t-s)B(\varepsilon k)/\varepsilon}Q(\varepsilon k)\,\hat{\Gamma}[u(s),u(s)]ds$$

$$\equiv \int_0^t F(t-s,k,\varepsilon k)\,\hat{\Gamma}[u(s),u(s)]ds$$

$$+ \frac{1}{\varepsilon}\int_0^t G((t-s)/\varepsilon,k,\varepsilon k)\,\hat{\Gamma}[u(s),u(s)]ds.$$

We put

$$(2.21)\qquad \begin{aligned} F[u,v](\varepsilon,t) &= \int_0^t F(t-s,k,\varepsilon k)\,\hat{\Gamma}[u(s),v(s)]ds,\\ G[u,v](\varepsilon,\tau) &= \int_0^\tau G\,(\tau-s,k,\varepsilon k)\,\hat{\Gamma}[u(\varepsilon s),v(\varepsilon s)]ds.\end{aligned}$$

Then we have the following

Lemma 2.5. *Let* $u_0 \in \dot{X}^{\alpha}_{\ell,\beta}$. *Then for any* $\gamma \geq 0$, $T \geq 0$, β' *and* m

$$F(t,k,\varepsilon k)u_0(k,\cdot) \in Z^{m,\alpha,\gamma}_{\ell,\beta',T}\ ,$$

$$G(\tau,k,\varepsilon k)u_0(k,\cdot) \in \mathcal{L}^{m,\alpha,\gamma,\sigma}_{\ell,\beta,T} \qquad (0 < \sigma < \sigma_0)\ ,$$

with

$$(2.22)\qquad \begin{aligned} \| F(t,k,\varepsilon k)u_0 \|_{(p,q),\alpha,\gamma,\ell,\beta',T} &\leq e_{(p,q)}(\beta',\beta)(1+T)^q\,|u_0|_{\alpha,\ell,\beta}\ .\\ \| G(\tau,k,\varepsilon k)u_0 \|_{(p,q),\alpha,\gamma,\sigma,\ell,\beta,T} &\leq g_{(p,q)}(\beta)\,|u_0|_{\alpha,\ell,\beta}\ .\end{aligned}$$

Lemma 2.6. *Let* $\alpha > 0$, $\gamma \geq 0$, $\ell > n$, $\beta \geq 0$ *and* $T > 0$. *with* $\alpha - \gamma T \geq 0$. *Let* $m = 0$ *or* $(0,1)$ *and* $m' = m + (0,1)$. *Then for* $u, v \in Z^{m,\alpha,\gamma}_{\ell,\beta,T}$ *or* $Z^{m,\alpha,\gamma,*}_{\ell,\beta,T}$ *and* $\tilde{u}, \tilde{v} \in Z^{m,\alpha,\gamma,\sigma}_{\ell,m,T}$,

$$(2.23) \qquad \| F[u,v] \|_{m,\alpha,\gamma,\ell,\beta',T} \leq b_m(\ell,\beta',\beta)\frac{1}{\gamma} \times \| u \|_{m,\alpha,\gamma,\ell,\beta,T} \| v \|_{m,\alpha,\gamma,\ell,\beta,T} .$$

$$(2.24) \qquad \| G[u,v] \|_{m,\alpha,\gamma,\ell,\beta,T} \leq b_m(\ell,\beta) \times \| u \|_{m,\alpha,\gamma,\ell,\beta,T} \| v \|_{m,\alpha,\gamma,\ell,\beta,T} ,$$

$$(2.25) \qquad \| G[\tilde{u},v] \|_{m,\alpha,\gamma,\sigma,\ell,\beta,T} \leq \tilde{b}_m(\ell,\beta,\sigma) \times \| \tilde{u} \|_{m,\alpha,\gamma,\sigma,\ell,\beta,T} \| v \|_{m,\alpha,\gamma,\ell,\beta,T} ,$$

$$(2.26) \qquad \| G[\tilde{u},\tilde{v}] \|_{m,\alpha,\gamma,\sigma,\ell,\beta,T} \leq \tilde{b}_m(\ell,\beta,\sigma) \times \| \tilde{u} \|_{m,\alpha,\gamma,\sigma,\ell,\beta,T} \| v \|_{m,\alpha,\gamma,\sigma,\ell,\beta,T},$$

where $\ell - |m| > n$, $\beta - |m| \geq 0$, $0 < \sigma < \sigma_0$, *For example, for* $m = 0$

$$b_m(\ell,\beta',\beta) = e(\beta',\beta)d(\ell,\beta) ,$$

$$b_m(\ell,\beta) = q(\beta)\{1 + \frac{1}{\sigma_0} g(\beta)c(\beta)\}d(\ell,\beta),$$

$$\tilde{b}_m(\ell,\beta,\sigma) = q(\beta)\{1 + \frac{1}{\sigma_0 - \sigma} g(\beta)c(\beta)\} \frac{\nu_0}{\nu_0 - \nu} d(\ell,\beta)$$

$q(\beta)$ = *the supremum of the norm of* $Q(k)$ *in the space* $\dot{L}^{\infty}_{\beta}$.

Moreover F *is continuous as a mapping*

$$Z^{m,\alpha,\gamma}_{\ell,\beta,T} \times Z^{m,\alpha,\gamma}_{\ell,\beta,T} \to Z^{m',\alpha,\beta}_{\ell,\beta,T} ,$$

$$Z^{m,\alpha,\gamma}_{\ell,\beta,T} \times Z^{m,\alpha,\gamma,*}_{\ell,\beta,T} \to Z^{m',\alpha,\gamma,*}_{\ell,\beta,T} .$$

G *is also continuous as a mapping*

$$Z^{m,\alpha,\gamma,*}_{\ell,\beta,T} \times Z^{m,\alpha,\gamma,*}_{\ell,\beta,T} \to Z^{m',\alpha,\gamma,*}_{\ell,\beta,T} ,$$