Modern Spectrum Analysis, II

Edited by
Stanislav B. Kesler

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Preface

THE past decade witnessed the emergence of the field of power spectrum estimation as a rather independent subfield of digital signal processing. Advances in very large scale integration technology have had a major impact on the technical areas to which spectrum estimation techniques are being applied. Research in power spectrum estimation has led to a variety of parametric and nonparametric techniques, extensions to multidimensional, multichannel, and spatio-temporal processing algorithms, and to computationally fast procedures. Concurrently, various techniques have been developed for estimation of signal parameters, like frequencies of sinusoids and poles and zeros. In some cases, parameter estimation was an intermediate step in some parametric spectrum estimation procedures, e.g., the determination of prediction error filter coefficients in the maximum entropy method. In general, however, parameter estimation techniques have grown to become a research area in its own right, although closely related to spectrum estimation. Extensive research has been conducted on the statistical properties of estimation techniques. The number of important applications have constantly been increased.

Rapid developments in the field have naturally led to an expansion of the technical literature. In 1978, in order to make a number of the important papers in the field easily accessible, the IEEE PRESS published the book, *Modern Spectrum Analysis*, edited by D. Childers [1]. Since the publication of that first reprint book, there have been many important developments in the field with regard to theoretical approaches, extensions, and applications. These have been particularly significant in the area of parameter estimation. As a consequence of such a rapid development, a number of publications entirely devoted to power spectrum estimation have appeared in print. They include the proceedings of four spectrum estimation workshops, initiated by the Rome Air Development Center [2], [3] and later sponsored by the IEEE Acoustics, Speech, and Signal Processing Society [4], [5]. A special issue of the *Proceedings of the IEEE* was published in 1982 [6], and one of the *IEE Proceedings*, part F, in 1983 [7]. These publications provide a chronology of spectral estimation research since 1978.

This second volume of selected reprints on power spectrum estimation complements the first. In selecting the papers to be included in this reprint book, an attempt was made to cover different topics. Each paper included contains at least one important aspect of spectral analysis not covered by other papers in the collection. On the other hand, every effort was made to reduce the inevitable overlap of the material to a minimum. In deciding among a number of papers treating a similar subject, we inclined toward selecting those which were the best combination of representation, length, and tutorial value. Therefore, in some cases, when a good review or tutorial paper was available, it was included in place of the original source paper.

Within the last ten years or so, an approximate time period covered in this collection, the papers on spectral analysis have appeared in a wide variety of technical journals published both within and outside the IEEE. Within the IEEE itself, the Acoustics, Speech, and Signal Processing Society has had the strongest interest in the area. However, a wide range of applications, as well as the interest in some theoretical aspects, have caused a significant number of papers in spectral analysis to appear in, at least, ten other IEEE Groups' and Societies' Publications. Therefore, it was possible to draw most of the selections for this reprint book from the IEEE sources, and still retain the generality in presenting the important topics. However, a number of important papers from other sources are also included.

Due to the page limitation, a number of excellent papers could not be included in spite of their appropriate content and/or tutorial value. In particular, most of the September 1982 special issue of the *Proceedings of the IEEE* [6] fits into this category. Therefore, it was decided not to include any paper which was published in [6], and treat that special issue as a companion volume.

The material in this book is divided into six parts. The division is somewhat arbitrary since a number of papers treat more than one specific topic. Part I, "Introduction", contains one paper, which became classic within a few years after being published. It gives a clear and thorough overview of up-to-date developments in the area of spectrum estimation. Due to the clear presentation of the subject matter, this paper represents excellent introductory reading for newcomers to the field. Part II, "Parametric Methods", deals with estimation procedures which are based on some *a priori* assumptions about the signal under analysis. The assumptions are made about the probabilistic mechanism (mostly, the second-order statistics) that governs the signal generating sources, resulting in assigning a particular parametric model to the signal. The most common models are autoregressive (AR), moving average (MA), and mixed autoregressive-moving average (ARMA) models. It is the discovery of fast computational methods in this class that initiated a rapid development of the field of spectrum estimation nearly two decades ago. Part III, "Nonparametric Methods", treats methods which do not assume *a priori* information about the signals. They are based on either the classical Fourier decomposition or some other orthogonal decomposition of signals. Since no model is imposed upon the signal, a method from the nonparametric class is theoretically, at least, applicable to a wider variety of signals than is a parametric method. However, the latter performs better on signals which *are known* to fit the assumed model.

Part IV, "Multidimensional, Multichannel, and Spatial Spectral Analysis", deals with processing of either the signals which are functions of more than one independent variable (multidimensional), or a set of signals which are vector processes of one independent variable (multichannel). Spatial-temporal spectral analysis of data received by an array of sensors is a typical example of the combination of the above cases. Array processing finds applications in virtually all areas where temporal-frequency spectral analysis is used. Parametric and nonparametric methods can be implemented using various computational algorithms. Some of them are given in Part V, "Algorithms and Adaptive Techniques". A majority of the earlier algorithms are based on the global optimization (e.g., in a least mean square sense) of some system parameters. These are recently being superceded by fast algorithms which are recursive and/or adaptive. Development of such algorithms enables faster processing of large amounts of input data. Important by-products of spectral analysis are signal detection and estimation of signal parameters. They are treated in Part VI, "Statistics and Detection", together with the statistical properties of the analysis techniques. An important issue in detection of multiple signals is the signal resolvability, and a few papers that deal with it are included in this part.

Since there are several papers in the companion volume [6] that should have been included in this reprint collection, we will list the contents of [6] with a brief description of each paper.

The first paper entitled "A historical perspective of spectrum estimation", by E. A. Robinson presents a detailed overview of the developments in spectral analysis from Newton to the present.

Nine of the remaining 12 papers that follow, are concerned with various aspects of parametric methods. J. A. Cadzow's paper, "Spectral estimation: An overdetermined rational model equation approach," and E. T. Jaynes' paper, "On the rationale of maximum-entropy methods," treat spectral analysis methods, based on rational modeling of time series, from two viewpoints. The former discusses parameter hypersensitivity when their estimates are obtained by using a minimal set of Yule–Walker equations, i.e., when the number of equations are equal to the number of parameters. It suggests counteracting this hypersensitivity by using more than minimal number of equations. The latter paper treats various methods from the information theoretic point of view.

The next paper, entitled "Maximum-entropy spectral analysis of radar clutter" by S. Haykin, B. W. Currie, and S. B. Kesler, describes the digital processor for classifying the different forms of radar clutter as encountered in an air traffic control environment. The Doppler-based features are obtained by the multisegment maximum-entropy procedure. The paper by J. P. Burg, D. G. Luenberger, and D. L. Wenger, entitled "Estimation of structured covariance matrices" approaches the subject through the maximum likelihood estimate of the covariance matrix, rather then the power spectrum density estimate. The underlying process is assumed to be zero-mean Gaussian, and the covariance matrix of the special structure is sought. When the maximum-entropy estimate is found from the covariance matrix obtained in this way, there is no line splitting effect.

In their paper, "Estimation of frequencies of multiple sinusoids: Making linear prediction perform like maximum likelihood", D. W. Tufts and R. Kumaresan suggest a modification of the least-squares linear prediction method, by replacing the usual covariance matrix estimate with the least squares approximation matrix having the lower rank. The estimation performance for short data records and low signal-to-noise ratio is significantly improved. This paper gives somewhat more detailed account of the subject than the paper [9] in Part II. Combination of forward and backward linear prediction is a part of a number of parametric spectral analysis procedures. It is naturally parametrized by lattice filter structures. A comprehensive summary of lattice algorithms for estimating model parameters is given in B. Friedlander's paper "Lattice methods for spectral estimation". It also shows the methods of computation of various model parameters from lattice parameters.

The equivalence between the problem of estimating the principal frequency components in the time series and the problem of determining the bearing of a radiating source with an array of sensors is presented in D. H. Johnson's paper "The application of spectral estimation methods to bearing estimation problems". This treatment is similar to the one given in the paper [6] in Part IV. J. H. McClellan's paper "Multidimensional spectral estimation" gives a thorough review of the subject and discusses several types of estimators including Fourier, MLM, MEM, and Pisarenko estimators.

The last paper in the section on parametric methods, entitled "Spectral approach to geophysical inversion by Lorentz, Fourier, and Radon transforms", written by E. A. Robinson, deals with the application of the spectral analysis to one-dimensional (1-D) and two-dimensional

(2-D) geophysical inversion problem. The objective of the inversion is to determine the structure of the earth from the seismic data obtained at the surface. The earth is modeled as being composed of horizontal layers with different propagation parameters, and the seismic raypaths are taken to be in either vertical (1-D) or slanted (2-D) direction.

First of the three papers in nonparametric section, ''Spectrum estimation and harmonic analysis'', by D. J. Thomson treats the estimation problem through the solution of an integral equation that defines a Fourier transform of the time series. The solution is given in terms of orthogonal data windows (discrete prolate spheroidal sequences). The method shows a good performance for time series with both narrow-band and wide-band spectral components. The following paper, entitled ''Robust-resistant spectrum estimation'', by R. D. Martin and D. J. Thomson deals with the preprocessing of time series which contain local perturbations, such as missing data points or non-Gaussian additive noise. These perturbations normally cause bias and variance increases in estimated spectra. To prevent this, authors suggest ''data cleaning'' by either one-sided or two-sided perturbation interpolators based on autoregressive approximations, prior to spectrum estimation.

The last paper in the nonparametric section, ''Spectral estimation using combined time and lag weighting'', by A. H. Nuttall and G. C. Carter, presents a computationally efficient spectral estimation method with good statistical properties. Also, their procedure yields classical methods, such as the Blackman–Tukey and the Welch method, as special cases. The procedure is described in the papers [4], [5] in Part III.

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[7] *IEE Proceedings*, part F, Special Issue on Spectral Analysis, vol. 130, part F, no. 3, Apr. 1983.

Contents

Part I
Introduction

THIS introductory section consists of a single paper, entitled "Spectrum analysis—A modern perspective" by S. M. Kay and S. L. Marple, Jr. This paper is a comprehensive review of the spectrum analysis techniques developed up to the time of its publication in 1981. In addition to the classical Blackman–Tukey and periodogram methods, the paper treats modern techniques by first examining the modeling and parameter identification approaches, on which these techniques are based. Modern techniques discussed include three rational modeling approaches namely, moving average (MA), autoregressive (AR), and autoregressive-moving average (ARMA), then the minimum variance distortionless unbiased technique (also known as Capon's maximum likelihood method), and the Pisarenko and Prony techniques. The comparative overview of all methods gives a significant tutorial value to the paper, so that it represents an excellent introductory reading for the newcomer to the field.

Spectrum Analysis—A Modern Perspective

STEVEN M. KAY, MEMBER, IEEE, AND STANLEY LAWRENCE MARPLE, JR., MEMBER, IEEE

Abstract—A summary of many of the new techniques developed in the last two decades for spectrum analysis of discrete time series is presented in this tutorial. An examination of the underlying time series model assumed by each technique serves as the common basis for understanding the differences among the various spectrum analysis approaches. Techniques discussed include the classical periodogram, classical Blackman-Tukey, autoregressive (maximum entropy), moving average, autoregressive-moving average, maximum likelihood, Prony, and Pisarenko methods. A summary table in the text provides a concise overview for all methods, including key references and appropriate equations for computation of each spectral estimate.

I. INTRODUCTION

ESTIMATION of the power spectral density (PSD), or simply the spectrum, of discretely sampled deterministic and stochastic processes is usually based on procedures employing the fast Fourier transform (FFT). This approach to spectrum analysis is computationally efficient and produces reasonable results for a large class of signal processes. In spite of these advantages, there are several inherent performance limitations of the FFT approach. The most prominent limitation is that of frequency resolution, i.e., the ability to distinguish the spectral responses of two or more signals. The frequency resolution in hertz is roughly the reciprocal of the time interval in seconds over which sampled data is available. A second limitation is due to the implicit windowing of the data that occurs when processing with the FFT. Windowing manifests itself as "leakage" in the spectral domain, i.e., energy in the main lobe of a spectral response "leaks" into the sidelobes, obscuring and distorting other spectral responses that are present. In fact, weak signal spectral responses can be masked by higher sidelobes from stronger spectral responses. Skillful selection of tapered data windows can reduce the sidelobe leakage, but always at the expense of reduced resolution.

These two performance limitations of the FFT approach are particularly troublesome when analyzing short data records. Short data records occur frequently in practice because many measured processes are brief in duration or have slowly time-varying spectra that may be considered constant only for short record lengths. In radar, for example, only a few data samples are available from each received radar pulse. In sonar, the motion of targets results in a time-varying spectral response due to Doppler effects.

In an attempt to alleviate the inherent limitations of the FFT approach, many alternative spectral estimation procedures have been proposed within the last decade. A comparison of the spectral estimates shown in Fig. 1 illustrate the improvement that may be obtained with nontraditional approaches. The three spectra illustrated were computed using the first nine autocorrelation lags[1] of a process consisting of two equi-amplitude sinusoids at 3 and 4 Hz in additive white noise. The conventional spectral estimate based on the nine known lags $R_{xx}(0), \cdots, R_{xx}(8)$ is shown in Fig. 1(a). The spectrum is a plot of 512 values obtained by application of a 512-point FFT to the nine lags, zero-padded with 503 zeros. This spectrum, often termed the Blackman-Tukey (BT) estimate of the PSD, is characterized by sidelobes, some of which produce negative values for the PSD, and by an inability to distinguish the two sinusoidal responses.

Fig. 1(b) shows the spectral response of the autoregressive (AR) method based on the same nine lags. The improvement in resolution over that shown in Fig. 1(a) has contributed to the popularity of this alternative spectral estimate. Although the AR spectral estimate was originally developed for geophysical data processing, where it was termed the maximum entropy method (MEM) [16], [37]-[39], [50], [84], [136], [138], [158], [221], [231], [246], [247], it has been used for applications in radar [75], [92], [99], [116], [125], [126], [216], sonar [122], [198], imaging [98], radio astronomy [162], [264], [265], biomedicine [71], [74], oceanography [96], ecological systems [88], and direction finding [70], [128], [233]. The AR approach to spectrum analysis is closely related to linear prediction coding (LPC) techniques used in speech processing [80], [130], [143], [145]. The AR PSD estimator fits an AR model to the data. The origin of AR models may be found in economic time series forecasting [31], [276] and statistical estimation [189]-[191]. The MEM approach makes different assumptions about the lags, but for practical purposes, the MEM and AR spectral estimators are identical for one-dimensional analysis of wide sense stationary, Gaussian processes.

The ultimate resolution of the two sinusoidal signals into two delta function responses in a uniform spectral floor, representing the white noise PSD level, is achieved with the Pisarenko harmonic decomposition (PHD) method shown in Fig. 1(c). This technique yields the most accurate estimate of the spectrum of sinusoids in noise, at least when the autocorrelation lags are known.

As evidenced by the spectrum examples of Fig. 1, the development of alternative spectral estimates in widely different application areas has led to a confusion of conflicting terminology and different algorithm development viewpoints. Thus

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[1] The autocorrelation function $R_{xx}(k)$ of a stochastic wide sense stationary discrete process x_n at lag k is defined in this paper as the expectation of the product $x_{n+k}x_n^*$, or $R_{xx}(k) = E[x_{n+k}x_n^*]$, where x_n is assumed to have zero mean. The * denotes complex conjugate, since complex processes are assumed in general, and $E(\ )$ denotes the expectation operator.

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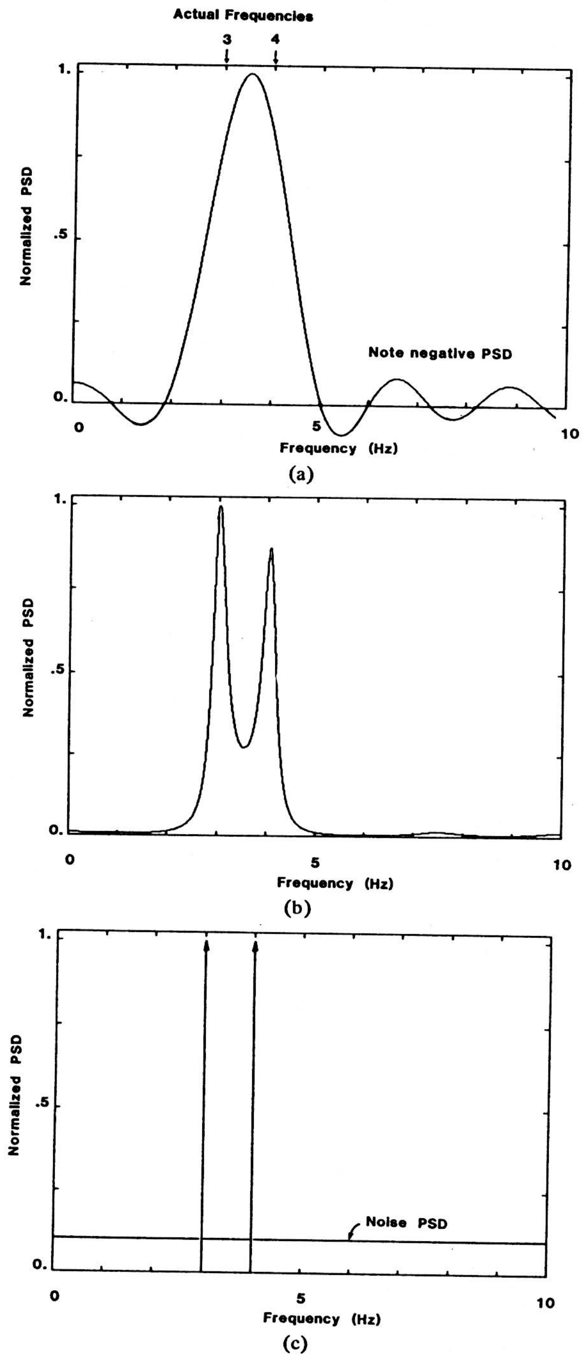

Fig. 1. Examples of three spectral estimates based on nine known autocorrelation lags of a process consisting of two equi-amplitude sinusoids in additive white noise (the variance of the noise is 10 percent of the sinusoid power). (a) BT PSD. (b) Autoregressive PSD. (c) Pisarenko harmonic decomposition PSD.

two purposes of this review are 1) to establish a common framework of terminology and symbols and 2) to unify the various approaches and algorithm developments that have evolved in various disciplines.

Claims have been made concerning the degree of improvement obtained in the spectral resolution and the signal detectability when AR and Pisarenko techniques are applied to sampled data [36], [206], [250], [251]. These performance advantages, though, strongly depend upon the signal-to-noise ratio (SNR), as might be expected. In fact, for low enough SNR's the modern spectral estimates are often no better than those obtained with conventional FFT processing [122], [150]. Even in those cases where improved spectral fidelity is achieved by use of an alternative spectral estimation procedure, the computational requirements of that alternative method may be significantly higher than FFT processing. This may make some modern spectral estimators unattractive for real-time implementation. Thus a third objective of this paper is to present tradeoffs among the various techniques. In particular, the performance advantages and disadvantages will be highlighted for each method, the computational complexity will be summarized, and criteria will be presented for determining if the selected spectral estimator is appropriate for the process being analyzed.

Some historical perspective is instructive for an appreciation of the basis for modern spectral estimation. The illustrious history of the Fourier transform can be traced back over 200 years [34], [223]. The advent of spectrum analysis based on Fourier analysis can be traced to Schuster, who was the first to coin the term "periodogram" [218], [219]. Schuster made a Fourier series fit to the variation in sun-spot numbers in an attempt to find "hidden periodicities" in the measured data. The next pioneering step was described in Norbert Wiener's classic paper on "generalized harmonic analysis" [269]. This work established the theoretical framework for the treatment of stochastic processes by using a Fourier transform approach. A major result was the introduction of the autocorrelation function of a random process and its Fourier transform relationship with the power spectral density. Khinchin [127] defined a similar relationship independently of Wiener.

Blackman and Tukey, in a classical publication in 1958 [25], provided a practical implementation of Wiener's autocorrelation approach to power spectrum estimation when using sampled data sequences. The method first estimates the autocorrelation lags from the measured data, windows (or tapers) the autocorrelation estimates in an appropriate manner, and then Fourier transforms the windowed lag estimates to obtain the PSD estimate. The BT approach was the most popular spectral estimation technique until the introduction of the FFT algorithm in 1965, generally credited to Cooley and Tukey [53]. This computationally efficient algorithm renewed an interest in the periodogram approach to PSD estimation. The periodogram spectral estimate is obtained as the squared magnitude of the output values from an FFT performed directly on the data set (data may be weighted). Currently, the periodogram is the most popular PSD estimator [17], [24], [32], [105]-[107], [109].

Conventional FFT spectral estimation is based on a Fourier series model of the data, that is, the process is assumed to be composed of a set of harmonically related sinusoids. Other time series models have been used in nonengineering fields for many years. Yule [276] and Walker [258] both used AR models to forecast trends in economic time series. Baron de Prony [202] devised a simple procedure for fitting exponential models to data obtained from an experiment in gas chemistry. Other models have arisen in the statistical and numerical analysis fields. The modern spectral estimators have their roots in these nonengineering fields of time series modeling.

The use of nontraditional spectral estimation techniques in a significant manner began in the 1960's. Parzen [189], in 1968, formally proposed AR spectral estimation. Independently in 1967, Burg [37] introduced the maximum entropy method, motivated by his work with linear prediction filtering in geoseismological applications. The one-dimensional MEM was shown formally by Van den Bos [255] to be equivalent to the AR PSD estimator. Prony's method also bears some mathematical similarities to the AR estimation algorithms. An area of current research is that of autoregressive-moving average (ARMA) models. The ARMA model is a generalization of the AR model. It appears that methods based upon these may provide even better resolution and performance than AR methods. The PHD [194], [195] is one example of a spectral estimation technique based upon a special case ARMA model.

The unifying approach employed in this paper is to view each spectral estimation technique as being based on the fitting of measured data to an assumed model. The variations in performance among the various spectral estimates may often be attributed to how well the assumed model matches the process under analysis [173]. Different models may yield similar results, but one may require fewer model parameters and is therefore more efficient in its representation of the process. Spectral estimates of various techniques computed from samples of a process consisting of sinusoids in colored Gaussian noise are presented in Section III to illustrate these variations. The process has both narrow-band and broad-band components. This process helps to illustrate how some spectral estimates tend to better estimate the narrow-band components while other spectral estimates better estimate the broad-band components of the spectra. This example process emphasizes the need to understand the underlying model before passing judgement on a spectral estimation method.

This tutorial is divided into five sections. Section II is the largest section. It contains a tutorial review of all the methods considered in this paper. Section III provides a summary table and illustration that highlights and compares the various modern spectral estimation methods. Section IV briefly examines other application areas that utilize the spectral estimation methods discussed in this paper.

A table of contents of these three sections is included below to enable the reader to quickly locate topics of interest.

No discussion of band-limited extrapolation techniques for spectral estimation is presented here since a good tutorial is already available [103]. The conclusion, Section V, makes observations concerning trends in research and application of modern spectral estimation.

II. Review of Spectral Estimation Techniques

A. Spectral Density Definitions and Basics

Traditional spectrum estimation, as currently implemented using the FFT, is characterized by many tradeoffs in an effort to produce statistically reliable spectral estimates. There are tradeoffs in windowing, time-domain averaging, and frequency-domain averaging of sampled data obtained from random processes in order to balance the needs to reduce sidelobes, to perform effective ensemble averaging, and to ensure adequate spectral resolution. To summarize the basics of conventional spectrum analysis, consider first the case of a deterministic analog waveform $x(t)$, that is a continuous function of time. For generality, $x(t)$ will be considered complex-valued in this paper. If $x(t)$ is absolute integrable, i.e., the signal energy $\mathcal{E}$ is finite

$$\mathcal{E} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty \tag{2.1}$$

then the continuous Fourier transform (CFT) $X(f)$ of $x(t)$ exists and is given by

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) \, dt. \tag{2.2}$$

(Note that (2.1) is a sufficient, but not a necessary condition for the existence of a Fourier transform [33].) The squared modulus of the Fourier transform is often termed the spectrum, $\mathcal{S}(f)$, of $x(t)$,

$$\mathcal{S}(f) = |X(f)|^2. \tag{2.3}$$

Parseval's energy theorem, expressed as

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df \tag{2.4}$$

is a statement of the conservation of energy; the energy of the time domain signal is equal to the energy of the frequency domain transform, $\int_{-\infty}^{\infty} \mathcal{S}(f) \, df$. Thus $\mathcal{S}(f)$ is an *energy spectral density* (ESD) in that it represents the distribution of energy as a function of frequency. If the signal $x(t)$ is sampled at equispaced intervals of Δt s to produce a discrete sequence $x_n = x(n\Delta t)$ for $-\infty < n < \infty$, then the sampled sequence can be represented as the product of the original time function $x(t)$ and an infinite set of equispaced Dirac delta functions $\delta(t)$. The Fourier transform of this product may be written, using distribution theory [33], as

$$X'(f) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(t)\delta(t - n\Delta t)\Delta t \right] \exp(-j2\pi ft) \, dt$$
$$= \Delta t \sum_{n=-\infty}^{\infty} x_n \exp(-j2\pi fn\Delta t). \tag{2.5}$$

Expression (2.5) corresponds to a rectangular integration approximation of (2.2); the factor Δt ensures conservation of integrated area between (2.2) and (2.5) as $\Delta t \to 0$. Expression (2.5) will be identical in value to the transform $X(f)$ of (2.2) over the interval $-1/(2\Delta t) \leqslant f \leqslant 1/(2\Delta t)$ Hz, as long as $x(t)$ is band limited and all frequency components are in this interval. Thus the continuous energy spectral density

$$\mathcal{S}'(f) = |X'(f)|^2 \tag{2.6}$$

for data sampled from a band-limited process is identical to that of (2.3).

If a) the data sequence is available from only a finite time window over $n = 0$ to $n = N - 1$, and b) the transform is discretized also for N values by taking samples at the frequencies $f = m\Delta f$ for $m = 0, 1, \cdots, N - 1$ where $\Delta f = 1/N\Delta t$, then one can develop the familiar discrete Fourier transform (DFT) [33] from (2.5),[2]

$$X_m = \Delta t \sum_{n=0}^{N-1} x_n \exp(-j2\pi m\Delta f n\Delta t)$$

$$= \Delta t \sum_{n=0}^{N-1} x_n \exp(-j2\pi mn/N)$$

$$\text{for } m = 0, \cdots, N-1. \tag{2.7}$$

Both (2.7) and its associated inverse transform are cyclic with period N. Thus by using (2.7), we have forced a periodic extension to both the discretized data and the discretized transform values, even though the original continuous data may not have been periodic. A discrete ESD may then be defined as

$$\mathcal{S}_m = |X_m|^2 \tag{2.8}$$

also for $0 \leqslant m \leqslant N - 1$. Both the discrete $\mathcal{S}_m$ and the continuous $\mathcal{S}'(f)$ have been termed *periodogram* spectral estimates. Note however that $\mathcal{S}_m$ and $\mathcal{S}'(f)$, when evaluated at $f = m/N\Delta t$ for $m = 0, \cdots, N - 1$, do not yield identical values. $\mathcal{S}_m$ is, in effect, a sampled version of a spectrum determined from the convolution of $X(f)$ with the transform of the rectangular window that contains the data samples. Thus the discrete spectrum $\mathcal{S}_m$ based on a finite data set is a distorted version of the continuous spectrum $\mathcal{S}'(f)$ based on an infinite data set.

A different viewpoint must be taken when the process $x(t)$ is a wide sense stationary, stochastic process rather than a deterministic, finite-energy waveform. The energy of such processes are usually infinite, so that the quantity of interest is the power (time average of energy) distribution with frequency. Also, integrals such as (2.2) normally do not exist for a stochastic process. For the case of stationary random processes, the autocorrelation function

$$R_{xx}(\tau) = E[x(t+\tau)x^*(t)] \tag{2.9}$$

provides the basis for spectrum analysis, rather than the random process $x(t)$ itself. The Wiener–Khinchin theorem relates $R_{xx}(\tau)$ via the Fourier transform to $\mathcal{P}(f)$, the PSD,

$$\mathcal{P}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) \exp(-j2\pi f\tau)\, d\tau. \tag{2.10}$$

[2] The inverse transform is given by $x_n = \Delta f \sum_{m=0}^{N-1} X_m \exp(+j2\pi mn/N)$ and the energy theorem is

$$\sum_{n=0}^{N-1} |x_n|^2 \Delta t = \sum_{n=0}^{N-1} |X_m|^2 \Delta f.$$

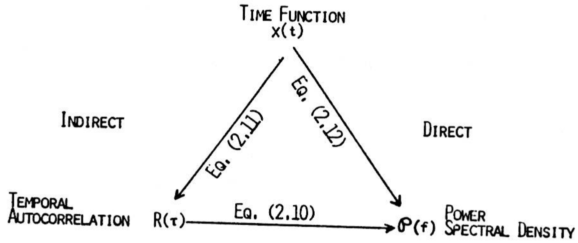

Fig. 2. Direct and indirect methods to obtain PSD (stationary and ergodic properties assumed).

As a practical matter, one does not usually know the statistical autocorrelation function. Thus an additional assumption often made is that the random process is ergodic in the first and second moments. This property permits the substitution of time averages for ensemble averages. For an ergodic process, then, the statistical autocorrelation function may be equated to

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau)x^*(t)\, dt. \tag{2.11}$$

It is possible to show [107], [132], [187], with the use of (2.11), that (2.10) may be equivalently expressed as

$$\mathcal{P}(f) = \lim_{T\to\infty} E\left\{\frac{1}{2T}\left|\int_{-T}^{T} x(t)\exp(-j2\pi ft)\, dt\right|^2\right\}. \tag{2.12}$$

The expectation operator is required since the ergodic property of $R_{xx}(\tau)$ does not couple through the Fourier transform; that is, the limit in (2.12) without the expected value does not converge in any statistical sense. Fig. 2 depicts the direct and indirect approaches to obtain the PSD from the signal $x(t)$, based on the formal relationships (2.10), (2.11), and (2.12).

Difficulties may arise if (2.12) is applied to finite data sets without regard to the expectation and limiting operations. Statistically inconsistent (unstable) estimates result if no statistical averaging is performed; i.e., the variance of the PSD estimate will not tend to zero as T increases without bound [183].

B. Traditional Methods

Two spectral estimation techniques based on Fourier transform operations have evolved. The PSD estimate based on the indirect approach via an autocorrelation estimate was popularized by Blackman and Tukey [14]. The other PSD estimate, based on the direct approach via an FFT operation on the data, is the one typically referred to as the periodogram.

With a finite data sequence, only a finite number of discrete autocorrelation function values, or lags, may by estimated. Blackman and Tukey proposed the spectral estimate

$$\hat{\mathcal{P}}_{\mathrm{BT}}(f) = \Delta t \sum_{n=-M}^{M} \hat{R}_{xx}(m) \exp(-j2\pi fm\Delta t) \tag{2.13}$$

based on the available autocorrelation lag estimates $\hat{R}_{xx}(m)$, where $-1/(2\Delta t) \leqslant f \leqslant 1/(2\Delta t)$ and ^ denotes an estimate. This spectral estimate is the discrete-time version of the Wiener–Khinchin expression (2.10). An obvious companion autocorrelation estimate, based on (2.11), is the unbiased estimator

$$\hat{R}_{xx}(m) = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x_{n+m}x_n^* \tag{2.14}$$