MELVIN LAX
WEI CAI
MIN XU

Random Processes in Physics and Finance

XFORD
INANCE

Random Processes in Physics and Finance

MELVIN LAX, WEI CAI, MIN XU

Department of Physics, City University of New York
Department of Physics, Fairfield University, Connecticut

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship, and education by publishing worldwide in

Oxford New York

Auckland Cape Town Dar es Salaam Hong Kong Karachi
Kuala Lumpur Madrid Melbourne Mexico City Nairobi
New Delhi Shanghai Taipei Toronto

With offices in

Argentina Austria Brazil Chile Czech Republic France Greece
Guatemala Hungary Italy Japan Poland Portugal Singapore
South Korea Switzerland Thailand Turkey Ukraine Vietnam

Published in the United States
by Oxford University Press Inc., New York

First published 2006

British Library Cataloguing in Publication Data
Data available

Library of Congress Cataloging in Publication Data
Data available

Printed in Great Britain
on acid-free paper by
Biddles Ltd., King's Lynn, Norfolk

ISBN 0-19-856776-6 978-0-19-856776-9

10 9 8 7 6 5 4 3 2 1

Random Processes in Physics and Finance

Preface

The name "Econophysics" has been used to denote the use of the mathematical techniques developed for the study of random processes in physical systems to applications in the economic and financial worlds. Since a substantial number of physicists are now employed in the financial arena or are doing research in this area, it is appropriate to give a course that emphasizes and relates physical applications to financial applications.

The course and text on *Random Processes in Physics and Finance* differs from mathematical texts by emphasizing the origins of noise, as opposed to an analysis of its transformation by linear and nonlinear devices. Of course, the latter enters any analysis of measurements, but it is not the focus of this work.

The text opens with a chapter-long review of probability theory to refresh those who have had an undergraduate course, and to establish a set of tools for those who have not. Of course, this chapter can be regarded as an oxymoron since probability includes random processes. But we restrict probability theory, in this chapter, to the study of random events, as opposed to random processes, the latter being a sequence of random events extended over a period of time.

It is intended, in this chapter, to raise the level of approach by demonstrating the usefulness of delta functions. If an optical experimenter does his work with lenses and mirrors, a theorist does it with delta functions and Green's functions. In the spirit of Mark Kac, we shall calculate the chi-squared distribution (important in statistical decision making) with delta functions. The normalization condition of the probability density in chi-square leads to a geometric result, namely, we can calculate the volume of a sphere in n dimensions without ever transferring to spherical coordinates.

The use of a delta function description permits us to sidestep the need for using Lebesgue measure and Stieltjes integrals, greatly simplifying the mathematical approach to random processes. The problems associated with Ito integrals used both by mathematicians and financial analysts will be mentioned below. The probability chapter includes a section on what we call the first and second laws of gambling.

Chapters 2 and 3 define random processes and provide examples of the most important ones: Gaussian and Markovian processes, the latter including Brownian motion. Chapter 4 provides the definition of a noise spectrum, and the Wiener–Khinchine theorem relating this spectrum to the autocorrelation. Our point of view here is to relate the abstract definition of spectrum to how a noise spectrum is measured.

Chapter 5 provides an introduction to thermal noise, which can be regarded as ubiquitous. This chapter includes a review of the experimental evidence, the thermodynamic derivation for Johnson noise, and the Nyquist derivation of the spectrum of thermal noise. The latter touches on the problem of how to handle zero-point noise in the quantum case. The zero-frequency Nyquist noise is shown to be precisely equivalent to the Einstein relation (between diffusion and mobility).

Chapter 6 provides an elementary introduction to shot noise, which is as ubiquitous as thermal noise. Shot noise is related to discrete random events, which, in general, are neither Gaussian nor Markovian.

Chapters 7–10 constitute the development of the tools of random processes.

Chapter 7 provides in its first section a summary of all results concerning the fluctuation–dissipation theorem needed to understand many aspects of noisy systems. The proof, which can be omitted for many readers, is a succinct one in density matrix language, with a review of the latter provided for those who wish to follow the proof.

Thermal noise and Gaussian noise sources combine to create a category of Markovian processes known as Fokker-Planck processes. A serious discussion of Fokker–Planck processes is presented in Chapter 8 that includes generation recombination processes, linearly damped processes, Doob's theorem, and multivariable processes.

Just as Fokker-Planck processes are a generalization of thermal noise, Langevin processes constitute a generalization of shot noise, and a detailed description is given in Chapter 9.

The Langevin treatment of the Fokker–Planck process and diffusion is given in Chapter 10. The form of our Langevin equation is different from the stochastic differential equation using Ito's calculus lemma. The transform of our Langevin equation obeys the ordinary calculus rule, hence, can be easily performed and some misleadings can be avoided. The origin of the difference between our approach and that using Ito's lemma comes from the different definitions of the stochastic integral.

Application of these tools contribute to the remainder of the book. These applications fall primarily into two categories: physical examples, and examples from finance. And these applications can be pursued independently.

The physical application that required learning all these techniques was the determination of the motion and noise (line-width) of self-sustained oscillators like lasers. When nonlinear terms are added to a linear system it usually adds background noise of the convolution type, but it does not create a sharp line. The question "Why is a laser line so narrow" (it can be as low as one cycle per second, even when the laser frequency is of the order of 10^{15} per second) is explained in Chapter 11. It is shown that autonomous oscillators (those with no absolute time origin) all behave like van der Pol oscillators, have narrow line–widths, and have a behavior near threshold that is calculated exactly.

Chapter 12 on noise in semiconductors (in homogeneous systems) can all be treated by the Lax–Onsager "regression theorem".

The random motion of particles in a turbid medium, due to multiple elastic scattering, obeys the classic Boltzmann transport equation. In Chapter 13, the center position and the diffusion behavior of an incident collimated beam into an infinite uniform turbid medium are derived using an elementary analysis of the random walk of photons in a turbid medium. In Chapter 14, the same problem is treated based on cumulant expansion. An analytical expression for cumulants (defined in Chapter 1) of the spatial distribution of particles at any angle and time, exact up to an arbitrarily high order, is derived in an infinite uniform scattering medium. Up to the second order, a Gaussian spatial distribution of solution of the Boltzmann transport equation is obtained, with exact average center and exact half-width with time.

Chapter 15 on the extraction of signals in a noisy, distorted environment has applications in physics, finance and many other fields. These problems are ill-posed and the solution is not unique. Methods for treating such problems are discussed.

Having developed the tools for dealing with physical systems, we learned that the Fokker–Planck process is the one used by Black and Scholes to calculate the value of options and derivatives. Although there are serious limitations to the Black–Scholes method, it created a revolution because there were no earlier methods to determine the values of options and derivatives. We shall see how hedging strategies that lead to a riskless portfolio have been developed based on the Black–Scholes ideas. Thus financial applications, such as arbitrage, based on this method are easy to handle after we have defined forward contracts, futures and put and call options in Chapter 16.

The finance literature expends a significant effort on teaching and using Ito integrals (integrals over the time of a stochastic process). This effort is easily circumvented by redefining the stochastic integral by a method that is correct for processes with nonzero correlation times, and then approaching the limit in which the correlation time goes to zero (the Brownian motion limit). The limiting result that follows from our iterative procedure, disagrees with the Ito definition of stochastic integral, and agrees with the Stratanovich definition. It is also less likely to be misleading as conflicting results were present in John Hull's book on *Options, Futures and Other Derivative Securities*.

In Chapter 17 we turn to methods that apply to economic time series and other forms including microwave devices and global warming. How can the spectrum of economic time series be evaluated to detect and separate seasonal and long term trends? Can one devise a trading strategy using this information?

How can one determine the presence of a long term trend such as global warming from climate statistics? Why are these results sensitive to the choice of year from solar year, sidereal year, equatorial year, etc. Which one is best? The most careful study of such time series by David J. Thomson will be reviewed. For example, studies of global warming are sensitive to whether one uses the solar year, sidereal year, the equatorial year or any of several additional choices!

This book is based on a course on *Random Processes in Physics and Finance* taught in the City College of City University of New York to students in physics who have had a first course in "Mathematical Methods". Students in engineering and economics who have had comparable mathematical training should also be capable of coping with the text. A review/summary is given of an undergraduate course in probability. This also includes an appendix on delta functions, and a fair number of examples involving discrete and continuous random variables.

A note from co-authors

Most parts of this book were written by Distinguished Professor Melvin Lax (1922–2002), originated from the class notes he taught at City University of New York from 1985 to 2001. During his last few years, Mel made a big effort in editing this book and, unfortunately, was not able to complete it before his untimely illness.

Our work on the book is mostly technical, including correcting misprints and errors in text and formulas, making minor revisions, and converting the book to LaTex. In addition, Wei Cai wrote Chapter 14, Section 10.3–10.5, Section 16.8, and made changes to Section 8.3, 16.4, 16.6 and 16.7; Min Xu wrote Chapter 13 and partly Section 15.6.

We dedicate our work in this book in memory of our mentor, colleague and friend Melvin Lax. We would like to thank our colleagues at the City College of New York, in particular, Professors Robert R. Alfano, Joseph L. Birman and Herman Z. Cummins, for their strong support for us to complete this book.

Wei Cai
Min Xu

Contents

A Note from Co-authors xiv

1 Review of probability 1
1.1 Meaning of probability 1
1.2 Distribution functions 4
1.3 Stochastic variables 5
1.4 Expectation values for single random variables 5
1.5 Characteristic functions and generating functions 7
1.6 Measures of dispersion 8
1.7 Joint events 12
1.8 Conditional probabilities and Bayes' theorem 16
1.9 Sums of random variables 19
1.10 Fitting of experimental observations 24
1.11 Multivariate normal distributions 29
1.12 The laws of gambling 32
1.13 Appendix A: The Dirac delta function 35
1.14 Appendix B: Solved problems 40

2 What is a random process 44
2.1 Multitime probability description 44
2.2 Conditional probabilities 44
2.3 Stationary, Gaussian and Markovian processes 45
2.4 The Chapman–Kolmogorov condition 46

3 Examples of Markovian processes 48
3.1 The Poisson process 48
3.2 The one dimensional random walk 50
3.3 Gambler's ruin 52
3.4 Diffusion processes and the Einstein relation 54
3.5 Brownian motion 56
3.6 Langevin theory of velocities in Brownian motion 57
3.7 Langevin theory of positions in Brownian motion 60
3.8 Chaos 64
3.9 Appendix A: Roots for the gambler's ruin problem 64
3.10 Appendix B: Gaussian random variables 66

4 Spectral measurement and correlation 69
4.1 Introduction: An approach to the spectrum of a stochastic process 69
4.2 The definitions of the noise spectrum 69
4.3 The Wiener–Khinchine theorem 71
4.4 Noise measurements 73
4.5 Evenness in ω of the noise? 75
4.6 Noise for nonstationary random variables 77
4.7 Appendix A: Complex variable notation 80

5 Thermal noise 82
5.1 Johnson noise 82
5.2 Equipartition 84
5.3 Thermodynamic derivation of Johnson noise 85
5.4 Nyquist's theorem 87
5.5 Nyquist noise and the Einstein relation 90
5.6 Frequency dependent diffusion constant 90

6 Shot noise 93
6.1 Definition of shot noise 93
6.2 Campbell's two theorems 95
6.3 The spectrum of filtered shot noise 98
6.4 Transit time effects 101
6.5 Electromagnetic theory of shot noise 104
6.6 Space charge limiting diode 106
6.7 Rice's generalization of Campbell's theorems 109

7 The fluctuation–dissipation theorem 113
7.1 Summary of ideas and results 113
7.2 Density operator equations 117
7.3 The response function 119
7.4 Equilibrium theorems 121
7.5 Hermiticity and time reversal 122
7.6 Application to a harmonic oscillator 123
7.7 A reservoir of harmonic oscillators 126

8 Generalized Fokker–Planck equation 129
8.1 Objectives 129
8.2 Drift vectors and diffusion coefficients 131
8.3 Average motion of a general random variable 134
8.4 The generalized Fokker–Planck equation 137
8.5 Generation–recombination (birth and death) process 139

8.6 The characteristic function 143
8.7 Path integral average 146
8.8 Linear damping and homogeneous noise 149
8.9 The backward equation 152
8.10 Extension to many variables 153
8.11 Time reversal in the linear case 160
8.12 Doob's theorem 162
8.13 A historical note and summary (M. Lax) 163
8.14 Appendix A: A method of solution of first order PDEs 164

9 Langevin processes 168
9.1 Simplicity of Langevin methods 168
9.2 Proof of delta correlation for Markovian processes 169
9.3 Homogeneous noise with linear damping 171
9.4 Conditional correlations 173
9.5 Generalized characteristic functions 175
9.6 Generalized shot noise 177
9.7 Systems possessing inertia 180

10 Langevin treatment of the Fokker–Planck process 182
10.1 Drift velocity 182
10.2 An example with an exact solution 184
10.3 Langevin equation for a general random variable 186
10.4 Comparison with Ito's calculus lemma 188
10.5 Extending to the multiple dimensional case 189
10.6 Means of products of random variables and noise source 191

11 The rotating wave van del Pol oscillator (RWVP) 194
11.1 Why is the laser line-width so narrow? 194
11.2 An oscillator with purely resistive nonlinearities 195
11.3 The diffusion coefficient 197
11.4 The van der Pol oscillator scaled to canonical form 199
11.5 Phase fluctuations in a resistive oscillator 200
11.6 Amplitude fluctuations 205
11.7 Fokker–Planck equation for RWVP 207
11.8 Eigenfunctions of the Fokker–Planck operator 208

12 Noise in homogeneous semiconductors 211
12.1 Density of states and statistics of free carriers 211
12.2 Conductivity fluctuations 215
12.3 Thermodynamic treatment of carrier fluctuations 216
12.4 General theory of concentration fluctuations 218

12.5 Influence of drift and diffusion on modulation noise 222

13 Random walk of light in turbid media 227
13.1 Introduction 227
13.2 Microscopic statistics in the direction space 229
13.3 The generalized Poisson distribution $p_n(t)$ 232
13.4 Macroscopic statistics 233

14 Analytical solution of the elastic transport equation 237
14.1 Introduction 237
14.2 Derivation of cumulants to an arbitrarily high order 238
14.3 Gaussian approximation of the distribution function 242
14.4 Improving cumulant solution of the transport equation 245

15 Signal extraction in presence of smoothing and noise 258
15.1 How to deal with ill-posed problems 258
15.2 Solution concepts 259
15.3 Methods of solution 261
15.4 Well-posed stochastic extensions of ill-posed processes 264
15.5 Shaw's improvement of Franklin's algorithm 266
15.6 Statistical regularization 268
15.7 Image restoration 270

16 Stochastic methods in investment decision 271
16.1 Forward contracts 271
16.2 Futures contracts 272
16.3 A variety of futures 273
16.4 A model for stock prices 274
16.5 The Ito's stochastic differential equation 278
16.6 Value of a forward contract on a stock 281
16.7 Black–Scholes differential equation 282
16.8 Discussion 283
16.9 Summary 286

17 Spectral analysis of economic time series 288
17.1 Overview 288
17.2 The Wiener–Khinchine and Wold theorems 291
17.3 Means, correlations and the Karhunen–Loeve theorem 293
17.4 Slepian functions 295
17.5 The discrete prolate spheroidal sequence 298
17.6 Overview of Thomson's procedure 300
17.7 High resolution results 301

17.8 Adaptive weighting 302
17.9 Trend removal and seasonal adjustment 303
17.10 Appendix A: The sampling theorem 303

Bibliography 307

Index 323

1
Review of probability

Introductory remarks

The purpose of this chapter is to provide a review of the concepts of probability for use in our later discussion of random processes. Students who have not had an undergraduate probability course may find it useful to have some collateral references to accompany our necessarily brief summary.

Bernstein (1998) provides a delightful historical popularization of the ideas of probability from the introduction of Arabic numbers, to the start of probability with de Mere's dice problem, to census statistics, to actuarial problems, and the use of probability in the assessment of risk in the stock market. Why was the book titled *Against the Gods* ? Because there was no need for probability in making decisions if actions are determined by the gods, it took the Renaissance period before the world was ready for probability.

An excellent recent undergraduate introduction to probability is given by Hamming (1991). The epic work of Feller (1957) is not, as its title suggests, an introduction, but a two-volume treatise on both the fundamentals and applications of probability theory. It includes a large number of interesting solved problems. A review of the basic ideas of probability is given by E. T. Jaynes (1958). A brief overview of the frequency ratio approach to probability of von Mises, the axiomatic approach of Kolmogorov, and the subjective approach of Jeffreys is presented below.

1.1 Meaning of probability

The definition of probability has been (and still is) the subject of controversy. We shall mention, briefly, three approaches.

1.1.1 Frequency ratio definition

R. von Mises (1937) introduced a definition based on the assumed existence of a limit of the ratio of successes S to the total number of trials N:

$$P_N = \frac{S}{N} = \frac{\#\text{ of successes}}{\#\text{ of trials}}. \tag{1.1}$$

If the limit exists:

$$\lim_{N\to\infty} P_N = P = \text{probability of a success}, \tag{1.2}$$

it is regarded as the definition of the probability of success. One can object that this definition is meaningless since the limit does not exist, in the ordinary sense, that for any ϵ there exists an N such that for all $M > N$, $|P_N - P| < \epsilon$. This limit will exist, however, in a probability sense; namely, the probability that these inequalities will fail can be made arbitrarily small. The Chebycheff inequality of Eq. (1.32) is an example of a proof that the probability of a deviation will become arbitrarily small for large deviations. What is the proper statement for the definition of probability obtained as a "limit" of ratios in a large series of trials?

1.1.2 A priori mathematical approach (Kolmogorov)

Kolmogorov (1950) introduced an axiomatic approach based on set theory. The Kolmogorov approach assumes that there is some fundamental set of events whose probabilities are known, e.g., the six sides of a die are assumed equally likely to appear on top. More complicated events, like those involving the tossing of a pair of dice, can be computed by rules of combining the more elementary events.

For variables that can take on continuous values, Kolmogorov introduces set theory and assigns to the probability, p, the ratio between the measure of the set of successful events and the measure of the set of all possible events. This is a formal procedure and begs the question of how to determine the elementary events that have equal probabilities. In statistical mechanics, for example, it is customary to assume a measure that is uniform in phase space. But this statement applies to phase space in Cartesian coordinates, not, for example in spherical coordinates. There is good reason, based on how discrete quantum states are distributed, to favor this choice. But there is no guide in the Kolmogorov approach to probability theory for making such a choice.

The rigorous axiomatic approach of Kolmogorov raised probability to the level of a fully acceptable branch of mathematics which we shall call mathematical probability. A major contribution to mathematical probability was made by Doob (1953) in his book on *Stochastic Processes* and his rigorous treatment of Brownian motion. But mathematical probability should be regarded as a subdivision of probability theory which includes consideration of how the underlying probabilities should be determined.

Because ideal Brownian motion involves white noise (a flat spectrum up to infinite frequencies) sample processes are continuous but not differentiable. This problem provides a stage on which mathematicians can display their virtuosity in set theory and Lebesgue integration. When Black and Scholes (1973) introduced a model for prices of stock in which the logarithm of the stock price executes a

Brownian motion, it supplied the first tool that could be used to price stock (and other) options. This resulted in a Nobel Prize, a movement of mathematicians (and physicists) into the area of mathematical finance theory and a series of books and courses in which business administration students were coerced into learning set and Lebesgue integration theory. This was believed necessary because integrals over Brownian motion variables could not be done by the traditional Riemann method as the limit of a sum of terms each of which is a product of a function evaluation and an interval. The difficulty is that with pure Brownian motion the result depends on where in the interval the function must be evaluated. Ito (1951) chose to define a stochastic integral by evaluating the function at the beginning of the interval. This was accompanied by a set of rules known as the Ito calculus.

Mathematical probability describes the rules of computation for compound events provided that the primitive probabilities are known. In discrete cases like the rolling of dice there are natural choices (like giving each side of the die equal probability). In the case of continuous variables, the choice is not always clear, and this leads to paradoxes. See for example Bertrand's paradox in Appendix B of this chapter. Feller (1957) therefore makes the logical choice of splitting his book into two volumes the first of which deals with discrete cases. The hard work of dealing with continuous variables is postponed until the second volume.

What "mathematical probability" omits is a discussion of how contact must be made with reality to determine a model that yields the correct measure for each set in the continuous case. The Ito model makes one arbitrary choice. Stratonovich (1963) chooses not the left hand point of the interval, but an average over the left and right hand points. These two procedures give different values to a stochastic integral. Both are arbitrary.

As a physicist, I (Lax) argue that white noise leads to difficulties because the integrated spectrum, or total energy, diverges. In a real system the spectrum can be nearly flat over a wide range but it must go to zero eventually to yield a finite energy. For real signals, first derivatives exist, the ordinary Riemann calculus works in the sense that the limiting result is insensitive to where in the interval the function is evaluated. Thus the Ito calculus can be avoided. One can obtain the correct evaluation at each stage, and then approach the limit in which the spectrum becomes flat at infinity. We shall see in Chapters 10 and 16 that this limiting result disagrees with Ito's and provides the appropriate result for the ideal Brownian limit.

1.1.3 Subjective probability

Jeffreys (1957) describes subjective probability in his book on *Scientific Inference*. One is forced in life to assign probabilities to events where the event may occur only once, so the frequency ratio can not be used. Also, there may be no obvious elementary events with equal probabilities, e.g. (1) what is the probability that